REMARK ON CERTAIN $C^*$-ALGEBRA EXTENSIONS
CONSIDERED BY G. SKANDALIS

ALEXANDER KAPLAN

(Communicated by Palle E. T. Jorgensen)

Abstract. Let $\Gamma$ be a nonamenable, discrete ICC subgroup of a connected simple Lie group of real-rank one. G. Skandalis established the exact sequence

$$0 \rightarrow K(l^2(\Gamma)) \rightarrow C^*(C^*_\lambda(\Gamma), C^*_\rho(\Gamma)) \rightarrow C^*_\lambda(\Gamma \times \Gamma) \rightarrow 0.$$

In this note we give sufficient conditions under which such a short exact sequence is not semi-split. In particular, we show that such an extension has no inverse in $\text{Ext}(C^*_\lambda(\Gamma \times \Gamma))$ provided that the $C^*$-algebra generated by the unitary representation $g \rightarrow \lambda(g)\rho(g) \otimes \lambda(g)\rho(g)$ of $\Gamma$ on $l^2(\Gamma) \otimes l^2(\Gamma)$ does not contain nonzero operators from the ideal $K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))$.

Let $G$ be a countable discrete group, and let $\lambda$ and $\rho$ denote the left- and right-regular representations of $G$ on $l^2(G)$, defined by

$$(\lambda(g)\xi)(t) = \xi(g^{-1}t) \quad \text{and} \quad (\rho(g)\xi)(t) = \xi(tg) \quad (\xi \in l^2(G)).$$

The $C^*$-algebras $C^*_\lambda(G)$ and $C^*_\rho(G)$, generated respectively by $\lambda(G)$ and $\rho(G)$, are commuting and $C^*_\rho(G) = JC^*_\lambda(G)J$, where $J$ is the involution on $l^2(G)$ defined by $(J\xi)(t) = \xi(t^{-1})$. Let $C^*_{\lambda,\rho}(G)$ denote the $C^*$-algebra generated by $C^*_\lambda(G) \cup C^*_\rho(G)$.

If $G$ satisfies the infinite conjugacy class (ICC) condition, then the map given by $\sum_i x_i \otimes y_i \mapsto \sum_i x_i J y_i J$ is an isomorphism of the algebraic tensor product $C^*_\lambda(G) \otimes C^*_\rho(G)$ onto a dense $^*$-subalgebra of $C^*_{\lambda,\rho}(G)$ [8, IV, 4.20]. This defines a $C^*$-norm $\| \|_{\lambda,\rho}$ on $C^*_{\lambda}(G) \otimes C^*_{\rho}(G)$ via $\| \sum_i x_i \otimes y_i \|_{\lambda,\rho} = \| \sum_i x_i J y_i J \|$; and thus $C^*_{\lambda,\rho}(G)$ is a faithful representation of the $C^*-$tensor product $C^*_\lambda(G) \otimes_{\lambda,\rho} C^*_\rho(G)$—the closure of $C^*_\lambda(G) \otimes C^*_\rho(G)$ in the norm $\| \|_{\lambda,\rho}$. It is known that the norm $\| \|_{\lambda,\rho}$ dominates the spatial (minimal) norm on $C^*_\lambda(G) \otimes C^*_\rho(G)$ (and is equal to the spatial norm when $G$ is amenable). Hence there is a canonical surjective $^*$-homomorphism $\varphi: C^*_{\lambda,\rho}(G) \rightarrow C^*_\lambda(G) \otimes_{\min} C^*_\rho(G) = C^*_\lambda(G \times G)$.

In [7] Skandalis showed that if $\Gamma$ is a nonamenable, discrete ICC subgroup of a connected simple Lie group of real-rank one, then $C^*_\lambda,\rho(\Gamma)$ contains $K(l^2(\Gamma))$, the algebra of compact operators on $l^2(\Gamma)$, and the following exact sequence
holds:

\[ (*) \quad 0 \to K(l^2(\Gamma)) \to C_\lambda^*(\Gamma) \xrightarrow{\varphi} C_\lambda^*(\Gamma \times \Gamma) \to 0. \]

This generalizes the earlier result of Akemann and Ostrand [1] obtained for \( \Gamma = F_2 \), the free group on two generators.

The following theorem provides a sufficient condition under which an extension \((*)\) has no inverse in \( \text{Ext}(C_\lambda^*(\Gamma \times \Gamma)) \).

**Theorem.** Let \( \Gamma \) be a discrete (nonamenable, ICC) group satisfying \((*)\). Suppose that for some \( g_i \in \Gamma \) and \( c_i \in \mathbb{C} \) \((i = 1, \ldots, N)\):

\( i \) the \( C^* \)-algebra \( C^*(\sum_{i=1}^n c_i \lambda(g_i) \rho(g_i)) \), generated by the operator \( \sum_{i=1}^n c_i \lambda(g_i) \rho(g_i) \otimes \lambda(g_i) \rho(g_i) \) on \( l^2(\Gamma) \otimes l^2(\Gamma) \), does not contain nonzero operators from the ideal \( K(l^2(\Gamma)) \otimes \mathcal{B}(l^2(\Gamma)) \otimes \mathcal{B}(l^2(\Gamma)) \) in \( \mathcal{B}(l^2(\Gamma)) \);

\( ii \) the norm \( \| \sum_{i=1}^n c_i \lambda(g_i \times g_i) \rho(g_i \times g_i) \| \) is not equal to the norm \( \| \sum_{i=1}^n c_i \lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i) \| \) of the operator \( \sum_{i=1}^n c_i \lambda(g_i \times g_i) \times \lambda(g_i \times g_i) \) on \( l^2(\Gamma \times \Gamma) \otimes l^2(\Gamma \times \Gamma) \). (In other words, for \( x = \sum_{i=1}^n c_i \lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i) \in C_\lambda^*(\Gamma \times \Gamma) \otimes C_\lambda^*(\Gamma \times \Gamma), \|x\|_{\min} \neq \|x\|_{\lambda,\rho} \).

Then the embedding of the algebra \( C_\lambda^*(\Gamma \times \Gamma) \) into the Calkin algebra \( \mathcal{B}(l^2(\Gamma))/K(l^2(\Gamma)) \), determined by \((*)\), does not have a completely positive lifting to \( \mathcal{B}(l^2(\Gamma)) \).

**Proof.** Let \( \sigma: C_\lambda^*(\Gamma \times \Gamma) \to \mathcal{B}(l^2(\Gamma))/K(l^2(\Gamma)) \) denote the embedding determined by \((*)\). Suppose \( \psi: C_\lambda^*(\Gamma \times \Gamma) \to \mathcal{B}(l^2(\Gamma)) \) is a completely positive lifting of \( \sigma \). Then we have the following commutative diagram:

\[
\begin{array}{c}
\mathcal{B}(l^2(\Gamma)) \\
\downarrow \psi \\
\mathcal{B}(l^2(\Gamma))/K(l^2(\Gamma))
\end{array}
\]

In particular, for \( g \in \Gamma \), \( \psi(\lambda(g \times g)) = \lambda(g)\rho(g) + k \) for some \( k \in K(l^2(\Gamma)) \) since \( \varphi(\lambda(g)\rho(g)) = \sigma(\lambda(g \times g)) \). We can assume that \( \psi \) is unital [2, Lemma 3.3]. From Stinespring's theorem, \( \psi(\cdot) = U^*\pi(\cdot)U \) for some representation \( \pi \) of \( C_\lambda^*(\Gamma \times \Gamma) \) on a Hilbert space \( \mathcal{H} \) and isometry \( U: l^2(\Gamma) \to \mathcal{H} \). Let \( \pi \otimes \pi \) denote the tensor product representation of \( C_\lambda^*(\Gamma \times \Gamma) \otimes_{\min} C_\lambda^*(\Gamma \times \Gamma) \) on \( \mathcal{H} \otimes \mathcal{H} \). Then the map \( \psi \otimes \psi: C_\lambda^*(\Gamma \times \Gamma) \otimes_{\min} C_\lambda^*(\Gamma \times \Gamma) \to C_\lambda^*(\Gamma \times \Gamma) \otimes_{\min} C_\lambda^*(\Gamma \times \Gamma) \), defined by \( \psi \otimes \psi(\cdot) = (U \otimes U)^*(\pi \otimes \pi)(\cdot)(U \otimes U) \), is contractive (completely positive) and satisfies

\[ \psi \otimes \psi(x \otimes y) = \psi(x) \otimes \psi(y) \quad \text{for all } x, y \in C_\lambda^*(\Gamma \times \Gamma). \]

Therefore, in particular,

\[ \psi \otimes \psi \left( \sum_{i=1}^n c_i \lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i) \right) = \sum_{i=1}^n c_i \lambda(g_i)\rho(g_i) \otimes \lambda(g_i)\rho(g_i) + j_0 \]

for some \( j_0 \in K(l^2(\Gamma)) \otimes \mathcal{B}(l^2(\Gamma)) + \mathcal{B}(l^2(\Gamma)) \otimes K(l^2(\Gamma)) \), and

\[ (1) \quad \left\| \sum_{i=1}^n c_i \lambda(g_i)\rho(g_i) \otimes \lambda(g_i)\rho(g_i) + j_0 \right\| \leq \left\| \sum_{i=1}^n c_i \lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i) \right\|. \]
Denote
\[ a = \sum_{i=1}^{n} c_i \lambda(g_i \times g_i) \rho(g_i \times g_i), \]
\[ b = \sum_{i=1}^{n} c_i \lambda(g_i) \rho(g_i) \otimes \lambda(g_i) \rho(g_i) \]
\[ \left( = \sum_{i=1}^{n} c_i (\lambda(g_i) \otimes \lambda(g_i))(\rho(g_i) \otimes \rho(g_i)) \right), \]
and
\[ J = K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma)). \]
The operators \( a \) and \( b \) are unitarily equivalent via the natural unitary transformation \( V: l^2(\Gamma) \otimes l^2(\Gamma) \to l^2(\Gamma \times \Gamma) \), and so are the \( C^* \)-algebras \( C^*(a) \) and \( C^*(b) \), generated respectively by \( a \) and \( b \). Since \( J \) is a closed two-sided ideal of \( B(l^2(\Gamma)) \otimes B(l^2(\Gamma)) \) and \( C^*(b) \cap J = (0) \), we can define a \( C^* \)-algebra homomorphism \( \theta: C^*(b) + J \to C^*(a) \) by
\[ \theta(z + j) = VzV^* \quad (z \in C^*(b), \ j \in J). \]
In particular, since \( \theta(b + j_0) = VbV^* = a \), we have \( \|a\| \leq \|b + j_0\| \). Combining this with (1), we get
\[ \left\| \sum_{i=1}^{n} c_i \lambda(g_i \times g_i) \rho(g_i \times g_i) \right\| \leq \left\| \sum_{i=1}^{n} c_i \lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i) \right\|. \]
Consequently, for \( x = \sum_{i=1}^{n} c_i \lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i) \in C_{\delta}^*(\Gamma \times \Gamma) \cap (K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))) = (0) \), we have \( \|x\|_{\lambda, \rho} \leq \|x\|_{\min} \). This implies \( \|x\|_{\lambda, \rho} = \|x\|_{\min} \), which contradicts the hypothesis.

**Corollary.** Let \( \Gamma \) be a discrete (nonamenable, ICC) group satisfying (\( \ast \)). Let \( \mu \) be the unitary representation of \( \Gamma \) on \( l^2(\Gamma) \otimes l^2(\Gamma) \) defined by \( \mu(g) = \lambda(g) \rho(g) \otimes \lambda(g) \rho(g) \), and let \( C_{\mu}^*(\Gamma) \) be the \( C^* \)-algebra generated by \( \mu \). If \( C_{\mu}^*(\Gamma) \cap (K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))) = (0) \), then the extension (class of) \( (\ast) \) has no inverse in \( \text{Ext}(C_{\lambda}^*(\Gamma \times \Gamma)) \).

**Proof.** The representation \( \mu \) contains the trivial representation of \( \Gamma \), since \( \mu(g)(\xi_0 \otimes \xi_0) = \xi_0 \otimes \xi_0 \) for all \( g \), where \( \xi_0 \) denotes a function supported by the identity of \( \Gamma \). On the other hand, the representation of \( \Gamma \) on \( l^2(\Gamma \times \Gamma) \), defined by \( \delta(g) = \lambda(g \times g) \otimes \lambda(g \times g) \), is equivalent to a multiple of the left-regular representation of \( \Gamma \) \([4, 13.11.3]\). Since \( \Gamma \) is nonamenable, the map \( \lambda(g \times g) \otimes \lambda(g \times g) \to \lambda(g) \rho(g) \otimes \lambda(g) \rho(g) \) does not extend to a \( \ast \)-homomorphism of \( C_{\delta}^*(\Gamma) \) onto \( C_{\mu}^*(\Gamma) \). Therefore, there are \( g_i \in \Gamma \) and \( c_i \in \mathbb{C} \) (\( i = 1, \ldots, n \)) such that
\[ \left\| \sum_{i=1}^{n} c_i \lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i) \right\| < \left\| \sum_{i=1}^{n} c_i \lambda(g_i) \rho(g_i) \otimes \lambda(g_i) \rho(g_i) \right\|. \]
As observed earlier,
\[ \left\| \sum_{i=1}^{n} c_i \lambda(g_i) \rho(g_i) \otimes \lambda(g_i) \rho(g_i) \right\| = \left\| \sum_{i=1}^{n} c_i \lambda(g_i \times g_i) \rho(g_i \times g_i) \right\|. \]
so that the conclusion follows from the preceding theorem.

**Remark.** The hypothesis of the corollary is reminiscent of, but is much weaker than, the inner amenability of $\Gamma \times \Gamma$ (see [5, 6, 3]). Let $\alpha$ and $\beta$ denote the unitary representations of $\Gamma \times \Gamma$ and $\Gamma$ on $l^2(\Gamma \times \Gamma)$, given by $\alpha(h) = \lambda(h) \rho(h)$ and $\beta(g) = \lambda(g \times g) \rho(g \times g)$. The representation $\mu$ of the corollary is unitarily equivalent to $\beta$; and $\beta = \alpha \circ \Delta$, where $\Delta$ denotes the diagonal map of $\Gamma$ into $\Gamma \times \Gamma$. If $\Gamma$ is an ICC group, the inner amenability of $\Gamma \times \Gamma$ is equivalent to $C_\mu^*(\Gamma) \cap K(l^2(\Gamma \times \Gamma)) = (0)$ [3]. In particular, in this case $C_\mu^*(\Gamma) \cap K(l^2(\Gamma) \otimes l^2(\Gamma)) = (0)$, and so also

$$C_\mu^*(\Gamma) \cap (K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))) = (0),$$

since by [9] $K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))$ is a proper ideal of $K(l^2(\Gamma) \otimes l^2(\Gamma))$.

**References**


**Department of Mathematics and Statistics, Wright State University, Dayton, Ohio 45435-0001**

E-mail address: akaplan@desire.wright.edu