THE NUMERICAL INDEX OF NONSELFADJOINT OPERATOR ALGEBRAS

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Abstract. Let $\mathfrak{A}$ be a (not necessarily selfadjoint) closed subalgebra of $B(H)$ that is the algebra of all bounded linear operators on a Hilbert space $H$. In this note, we prove that the range of numerical index of $\mathfrak{A}$ as an algebra is the whole of the interval $[\frac{1}{2}, 1]$.

Let $H$ be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on $H$. Given $T \in B(H)$, we define the numerical range $W(T)$ and the numerical radius $w(T)$ as

$$W(T) = \{(Tx, x) : x \in H, \|x\| = 1\},$$

$$w(T) = \sup\{\|\lambda\| : \lambda \in W(T)\}.$$ Let $\mathfrak{A}$ be a closed (not necessarily selfadjoint) subalgebra of $B(H)$. As in [1, 2], we recall the numerical index $n(\mathfrak{A})$ of $\mathfrak{A}$ is defined by

$$n(\mathfrak{A}) = \inf\{w(T) : T \in \mathfrak{A}, \|T\| = 1\}.$$ Since $\frac{1}{2}\|T\| \leq w(T) \leq \|T\|$ for every $T \in B(H)$, we have $\frac{1}{2} \leq n(\mathfrak{A}) \leq 1$. As in [3], Crabb, Duncan, and McGregor showed that if $\mathfrak{A}$ is selfadjoint, that is, $\mathfrak{A}$ is a $C^*$-algebra, then $n(\mathfrak{A}) = 1$ or $\frac{1}{2}$ if $\mathfrak{A}$ is commutative or not commutative, respectively. In this note, we shall study the numerical index of nonselfadjoint operator algebras. As a result, we shall prove that the range of numerical index of nonselfadjoint operator algebra is the whole of the closed interval $[\frac{1}{2}, 1]$. That is, we have the following theorem.

Theorem. For every real number $r$ such that $\frac{1}{2} \leq r \leq 1$, there exists a Hilbert space $H$ and a (nonselfadjoint) closed subalgebra $\mathfrak{A}$ of $B(H)$ such that $n(\mathfrak{A}) = r$.

Proof. Let $r$ be a real number such that $\frac{1}{2} \leq r \leq 1$. Let $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and let $B$ be a normal operator on a Hilbert space $K$ such that the spectrum $\sigma(B)$ of $B$ is $\{\lambda \in \mathbb{C} : |\lambda| = r\}$. Put $T = A \oplus B$ on $H = \mathbb{C}^2 \oplus K$. We now consider the closed subalgebra $\mathfrak{A}_T$ generated by $T$ and $I$, that is, $\mathfrak{A}_T$ is the norm closure

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of \{p(T) : p \text{ is polynomial}\}. Since \(T\) is the convexoid operator such that \(\|T\| = 1\) and \(w(T) = r\), we have \(n(\mathfrak{A}_T) \leq r\). Therefore, to prove \(n(\mathfrak{A}_T) \geq r\), it is sufficient to prove that \(w(p(T)) \geq r\|p(T)\|\) for every polynomial \(p\), by the continuity of numerical radius.

For every polynomial \(p\), we now have \(w(p(T)) = \max\{w(p(A)), w(p(B))\}\), \(\|p(T)\| = \max\{|\|p(A)\|, |\|p(B)\||\}\), and \(w(p(B)) = \|\|p(B)\||\) = \(\max_{|z|=1} |p(z)|\). Putting \(p(z) = \sum_{n=0}^m a_n z^n\), it is clear that

\[
w(p(A)) = w(a_0 I + a_1 A) = |a_0| + \frac{1}{2}|a_1|
\]

and

\[
\|p(A)\| = \left\| \begin{bmatrix} a_0 & 0 \\ a_1 & a_0 \end{bmatrix} \right\| = \left( |a_0|^2 + \frac{1}{4}|a_1|^2 \right)^{1/2} + \frac{1}{2}|a_1|.
\]

If \(\|p(A)\| \leq \|p(B)\|\) then \(w(p(A)) \leq \|p(B)\|\), and so \(w(p(T)) = \|p(B)\| = \|p(T)\|\). If \(\|p(A)\| > \|p(B)\|\), then we have \(w(p(T)) \geq \|p(B)\|\) and \(\|p(T)\| = \|p(A)\|\). Since

\[
\|p(B)\| = \max_{|z| \leq r} |p(z)| = \max_{|z| \leq 1} |a_0 + a_1 r z + a_2 r^2 z^2 + \cdots + a_m r^m z^m|,
\]

by the Carathéodory-Fejér Theorem (cf. [9, 2.5]), we have

\[
\|p(B)\| \geq \left\| \begin{bmatrix} a_0 & 0 \\ a_1 r & a_0 \end{bmatrix} \right\| = \left( |a_0|^2 + \frac{r^2}{4}|a_1|^2 \right)^{1/2} + \frac{r}{2}|a_1|.
\]

Therefore, we have

\[
w(p(T)) \geq \|p(B)\| \geq \left( |a_0|^2 + \frac{r^2}{4}|a_1|^2 \right)^{1/2} + \frac{r}{2}|a_1|
\]

\[
\geq \left( r^2 |a_0|^2 + \frac{r^2}{4}|a_1|^2 \right)^{1/2} + \frac{r}{2}|a_1|
\]

\[
\geq r \left( \left( |a_0|^2 + \frac{1}{4}|a_1|^2 \right)^{1/2} + \frac{1}{2}|a_1| \right)
\]

\[
= r\|p(A)\| = r\|p(T)\|.
\]

This implies that \(n(\mathfrak{A}_T) = r\). This completes the proof.

Remark 1. We recall that the numerical index of a normed algebra \(\mathfrak{A}\) is defined by

\[
n(\mathfrak{A}) = \inf\{v(a) : a \in \mathfrak{A}, \|a\| = 1\},
\]

where \(v(a)\) is the numerical radius of \(a\) (cf. [1, 2]). Then \(e^{-1} \leq n(\mathfrak{A}) \leq 1\). In this case, the range of the numerical index is the whole of the closed interval \([e^{-1}, 1]\).

Remark 2. Take \(T \in B(H)\) and consider the norm closed subalgebra \(\mathfrak{A}_T\) of \(B(H)\) generated by \(T\) and \(I\). Then we have an interest in the class of operators satisfying \(n(\mathfrak{A}_T) = r\), in particular, \(n(\mathfrak{A}_T) = 1\). As in the proof of Theorem, \(n(\mathfrak{A}_T) = 1\) if and only if \(\|p(T)\| = w(p(T))\) for every polynomial \(p\). In [4, III], Fujii defined the notion of the class of operators, that is, an operator \(T \in B(H)\) is hen-spectroid if \(\sigma(T)\) is the spectral set for \(T\), where \(\sigma(T)\) is the complement of the unbounded component of the complement of the spectrum \(\sigma(T)\) of \(T\). By [4, III, Theorem 13], an operator \(T \in B(H)\) satisfies \(n(\mathfrak{A}_T) = 1\) if and only if \(T\) is hen-spectroid.
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