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ON THE ZEROS OF THE SOLUTIONS OF $y'' + P(z)y = 0$ WHERE $P(z)$ IS A POLYNOMIAL

LI-CHIEN SHEN

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ABSTRACT. Let $\{z_n\}$ be the nonzero zeros of the differential equation $y'' + P(z)y = 0$, where $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_N z^N$, and let $c_k = \sum_{n=1}^{\infty} 1/z_n^k$ for $k \geq [N/2] + 2$. We show that c_k is a rational function of a_n, $n = 0, 1, 2, \ldots, N$; futhermore, the successive c_k can be computed from previous c_k's by a simple recurrence relation.

The main purpose of this note is to study the zeros of the solutions of the differential equation

$$y'' + P(z)y = 0, \tag{1}$$

where

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_N z^N, \qquad a_N \neq 0. \tag{2}$$

To motivate and focus our discussion, we first consider the following seemingly unrelated example. Let $\zeta(z) = \sum_{n=1}^{\infty} 1/n^z$ be the zeta function of Riemann. One of the most interesting properties of the zeta function is the following well-known recurrence relation among $\zeta(2k)$, $k = 1, 2, 3, \ldots$.

Theorem A. *$\zeta(2) = \pi^2/6$ and for $k \geq 2$*

$$(2k+1)\zeta(2k) = 2\sum_{m=1}^{k-1} \zeta(2m)\zeta(2k-2m). \tag{3}$$

The methods to derive the above identity are, indeed, numerous. In this note, we shall approach it from the standpoint of differential equations. To this end, we observe that $\sin \pi z$ is a solution of the differential equation $y'' + \pi^2 y = 0$ and its zeros are precisely all the integers. Therefore, one interpretation of the identity (3) is that the sums of the (nonzero) zeros of a solution of the differential equation $y'' + \pi^2 y = 0$ raised to even powers $-2k$ satisfy a simple rational recurrence relation (3). And it turns out, as we shall see in Theorem 1, that identities similar to (3) hold, in general, for the sums of the reciprocals of the (nonzero) zeros (of a solution of the differential equation (1)) raised to certain powers.

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Since the natures of the solutions of (1) have been investigated extensively, we refer the readers to the book by Hille [2]. In this paper, we only need the following elementary fact.

Theorem B. *Let f be a solution of the differential equation* (1) *and let* $\{z_n\}$ *be the nonzero zeros of* f. *Then*

(i) f *is an entire function of order* $N/2+1$;

(ii) *all the zeros of* f *are simple* (*i.e., of multiplicity one*);

(iii) f *can be represented as*

$$f(z) = Az^a e^{Q(z)} \prod \left(1 - \frac{z}{z_n}\right) e^{z/z_n + (z/z_n)^2/2 + \cdots + (z/z_n)^p/p}, \tag{4}$$

where $a = 0$ *or* 1, A *is a constant, and*

$$Q(z) = b_1 z + b_2 z^2 + b_3 z^3 + \cdots + b_p z^p \tag{5}$$

with

$$p = \begin{cases} N/2+1 & \text{if } N \text{ is even}, \\ (N+1)/2 & \text{if } N \text{ is odd}. \end{cases}$$

We remark that when N is even, $\{z_n\}$ can be a finite set or an empty set, so, in these cases, the product in (4) is either a finite product or identically 1. We also order $\{z_n\}$ so that

$$|z_1| \le |z_2| \le |z_3| \le \cdots \le |z_n| \le \cdots.$$

Now differentiating (4) logarithmically and expanding the summands in terms of the power series at $z = 0$, we obtain

(6)

$$\frac{f'}{f}(z) = \frac{1}{z}\left(a + b_1 z + 2b_2 z^2 + \cdots + pb_p z^p - \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left(\frac{1}{z_n}\right)^{m+p+1} z^{m+p+1}\right)$$

for $|z| < |z_1|$.

We now define $c_0 = -a$ and

$$c_k = \begin{cases} -kb_k & \text{for } 1 \le k \le p, \\ \sum_{n=1}^{\infty} (1/z_n)^k & \text{for } k \ge p+1. \end{cases}$$

Then (6) becomes

$$\frac{f'}{f}(z) = -\sum_{k=0}^{\infty} c_k z^{k-1}. \tag{7}$$

Thus

$$\left(\frac{f'}{f}\right)'(z) = -\sum_{k=0}^{\infty} (k-1) c_k z^{k-2} \tag{8}$$

and

$$\left(\frac{f'}{f}\right)^2(z) = \frac{1}{z^2} \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} c_m c_{k-m}\right) z^k. \tag{9}$$

From (1), we obtain

$$\left(\frac{f'}{f}\right)' = \frac{f''}{f} - \left(\frac{f'}{f}\right)^2 = -P(z) - \left(\frac{f'}{f}\right)^2. \tag{10}$$

Substituting(8) and (9) into (10) and equating the coefficients of the power series on both sides, we derive

Theorem 1. *The coeffcients* c_n, $n = 0, 1, 2 \ldots$, *in* (7) *can be computed successively by the algorithm*

$$c_0 c_1 = 0, \tag{11}$$

$$(k-1)c_k = a_{k-2} + \sum_{m=0}^{k} c_m c_{k-m} \quad \text{for } 2 \le k \le N+2, \tag{12}$$

and

$$(k-1)c_k = \sum_{m=0}^{k} c_m c_{k-m} \quad \text{for } k \ge N+3. \tag{13}$$

In particular, the coefficients of the polynomial $Q(z)$ in (5) can be computed from the following

Corollary 1. $ab_1 = 0$, *and for* $2 \le k \le p$

$$(k-1)kb_k = a_{k-2} + \sum_{m=1}^{k-1} m(k-m)b_m b_{k-m}.$$

And choosing $k = p+1$ in (12), we have

Corollary 2.

$$\sum_{n=1}^{\infty} \frac{1}{z_n^{p+1}} = \frac{1}{2a+p}\left(a_{p-1} + \sum_{m=1}^{p} m(p+1-m)b_m\, b_{p+1-m}\right).$$

Clearly we see that the sum of all the nonzero zeros of f raised to $-(p+1)$th power is a rational function of a_n, $0 \le n \le p-1$. And Theorem A follows immediately from Corollary 2 and (13) by observing that in this case we have $p = 1$, $P(z) = \pi^2 = a_0$, and $b_1 = 0$.

Since it is rare for a sequence of numbers to satisfy any regular patterns among them, hence for $\{c_n\}$ to satisfy simple rational recurrence relations, the zeros of the solutions of the differential equation (1) must be distributed in a very special and magical manner.

It is interesting to note that the above method can also be used to obtain the exact sums of the reciprocals of the (nonzero) zeros of various orthogonal polynomials satisfying linear second-order differential equations. We illustrate this by considering the Legendre polynomials $P_n(z)$. It is known that they satisfy the differential equation

$$(1-z^2)y'' - 2zy' + n(n+1)y = 0. \tag{14}$$

From this, we obtain

$$\left(\frac{y'}{y}\right)' = \frac{y''}{y} - \left(\frac{y'}{y}\right)^2 = \frac{2z}{1-z^2}\left(\frac{y'}{y}\right) - \left(\frac{y'}{y}\right)^2 - \frac{n(n+1)}{1-z^2}. \tag{15}$$

We now recall that $P_{2n}(z)$ is an even function, $P_{2n}(0) \neq 0$, and $P_{2n+1}(z)$ is odd. Let $\{x_{n,i}\}$ be the nonzero zeros of $P_n(z)$. Then

$$P_n(z) = A_n z^{r(n)} \prod (z - x_{n,i}),$$

where

$$r(n) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

and A_n is a constant.

Differentiating $P_n(z)$ logarithmically, we obtain

$$\frac{P_n'}{P_n} = -\frac{1}{z}\sum_{k=0}^{\infty} S_n(2k) z^{2k}, \tag{16}$$

where $S_n(0) = -r(n)$ and $S_n(k) = \sum x_{n,i}^{-k}$. We observe that since $P_n(z)$ is either even or odd, $S_n(2k+1) = 0$ for all $k = 0, 1, 2, \dots$.

Substituting (16) into (15) and repeating the same argument as we did before for Theorem 1, we obtain the following recurrence relation for the sums of the reciprocals of the (nonzero) zeros of the Legendre polynomials raised to even powers:

$$S_n(2) = \frac{n(n+1) - 2r(n)}{1 + 2r(n)}$$

and for $k \geq 1$

$$\begin{aligned} &(2k + 1 + 2r(n))\, S_n(2k+2) \\ &\quad = (n(n+1) - 2r(n)) + 2\sum_{j=1}^{k} S_n(2j) + S_n(2j) S_n(2k + 2 - 2j). \end{aligned}$$

Therefore, when n is even, we have

$$S_n(2) = n(n+1), \quad S_n(4) = \tfrac{1}{3} n(n+1)(n^2 + n + 3), \ \dots, \ \text{etc.}$$

Many other closely related results can also be found in [1] and in the references therein.

We conclude this short note with several comments that will be helpful in understanding the essence of Theorem 1. We first note that by choosing $f(z) = \sin \pi z$, (10) becomes the familiar trigonometric identity $1 + \cot^2 z = \csc^2 z$. Theorem A is derived from this identity by equating the corresponding coefficients of their power series. Thus, we can view Theorem A as a reformulation of the above trigonometric identity in terms of the values of Riemann zeta function at the even integers. The solutions of the differential equation (10) are often regarded as generalizations of the sine and cosine functions. In [2] as well as in the comprehensive monograph [4] on the differential equation (1), the asymptotic formulas for the zeros of its solutions in various sectors in the complex plane are derived. Indeed, in many ways the solutions share a great deal in common with the sine and cosine functions. Thus the identity (10) can be view as an analogue of the trigonometric identity $1 + \cot^2 z = \csc^2 z$. With this in mind, it is, perhaps, not surprising that the sums of the reciprocals of the zeros of differential equation (1) also satisfy a simple recurrence relation. However, if one attempts to generalize further by considering $y'' + A(z)y = 0$, where A is entire, then the following result shows that it is most unlikely that the zeros of its solutions will demonstrate any simple behavior.

Theorem C [3]. *Given any two disjoint point sets S_1 and S_2 in the complex plane with no finite limit points, then there exists an entire function $A(z)$ such that the differential equation $y'' + A(z)y = 0$ possesses two linearly independent solutions $W_1(z)$ and $W_2(z)$ with their zeros precisely at S_1 and S_2, respectively.*

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Department of Mathematics, University of Florida, Gainesville, Florida 32611-2082