PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 117, Number 4, April 1993

NORMAL SPACES WHOSE STONE-ČECH REMAINDERS HAVE COUNTABLE TIGHTNESS

JIN-YUAN ZHOU

(Communicated by Franklin D. Tall)

ABSTRACT. We prove, assuming PFA, that each normal space whose Stone-Čech remainder has countable tightness is ACRIN. A normal space X is called ACRIN if each of its regular images is normal. Fleissner and Levy proved that if X is normal and every countably compact subset of the Stone-Čech remainder $\beta X \setminus X$ is closed in $\beta X \setminus X$, then X is ACRIN. They asked if each normal space whose Stone-Čech remainder has countable tightness is ACRIN. Theorem 2 gives the positive answer assuming the Proper Forcing Axiom.

It is well known that the tightness of $\beta\omega\setminus\omega$ is 2^{ω}. Since every not countably compact Hausdorff space contains a closed copy of ω, the next lemma is easy to prove.

Lemma 1. *If X is a normal space and $\beta X \setminus X$ has countable tightness, then X is countably compact.*

Theorem 2 (PFA). *If X is a normal space and $\beta X \setminus X$ has countable tightness, then X is ACRIN.*

Proof. Let $f\colon X \to Y$ be a continuous map and Y be regular. We prove that Y is normal. By virtue of Lemma 5 of [FL] there exist Z and bf such that $X \subseteq Z \subseteq \beta X$ and bf is a perfect map from Z onto Y with $bf|_X = f$. Since $\beta X \setminus X$ has countable tightness, it is easy to see that the spaces X, Y, and Z are all countably compact. Since perfect mappings preserve normality, we only need to prove that Z is normal. Let K and L be two disjoint closed subsets of Z. We will prove that $\overline{K}^{\beta X} \cap \overline{L}^{\beta X} = \varnothing$. Since X is normal, we have $\overline{K \cap X}^{\beta X} \cap \overline{L \cap X}^{\beta X} = \varnothing$. Take open subsets U and V of βX such that $\overline{U}^{\beta X} \cap \overline{V}^{\beta X} = \varnothing$, $\overline{K \cap X}^{\beta X} \subseteq U$, and $\overline{L \cap X}^{\beta X} \subseteq V$. Let $U' = U \setminus \overline{L}^{\beta x}$, $V' = V \setminus \overline{K}^{\beta X}$, and $T = (\overline{K \cup L}^{\beta X}) \setminus (U' \cup V')$. Obviously T is a closed subset of βX and is contained in $\beta X \setminus X$. Thus T is a compact space of countable tightness. Furthermore, $K \setminus U$ and $L \setminus V$ are contained in T and are closed subsets of the countably compact space Z. We have proved that $K \setminus U$ and $L \setminus V$ are countably compact subsets of a compact space of countable tightness. By virtue of Balogh's Theorem [Ba, 2.1], $K \setminus U$ and $L \setminus V$ are compact. Thus

Received by the editors November 23, 1990 and, in revised form, July 18, 1991.
1991 *Mathematics Subject Classification.* Primary 54C05, 54D15, 54D40.

we have

$$\begin{aligned}\overline{K}^{\beta X} \cap \overline{L}^{\beta X} &= (\overline{K \cap U}^{\beta X} \cup (K \setminus U)) \cap (\overline{L \cap V}^{\beta X} \cup (L \setminus V)) \\ &= (\overline{K \cap U}^{\beta X} \cap (L \setminus V)) \cup (\overline{L \cap V}^{\beta X} \cap (K \setminus U)) \\ &\subseteq (\overline{K}^{\beta X} \cap L) \cup (\overline{L}^{\beta X} \cap K) = \varnothing .\end{aligned}$$

We are done.

Acknowledgment

We acknowledge the referee's help in writing this paper.

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Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, People's Republic of China

Current address: Department of Mathematic and Statistics, York University, 4700 Keele Street, North York, Ontario, Canada M3J 1P3

E-mail address: ysma9202@clid.yorku.ca