NORMAL SPACES WHOSE STONE-ČECH REMAINDERS HAVE COUNTABLE TIGHTNESS

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Abstract. We prove, assuming PFA, that each normal space whose Stone-Čech remainder has countable tightness is ACRIN. A normal space $X$ is called ACRIN if each of its regular images is normal. Fleissner and Levy proved that if $X$ is normal and every countably compact subset of the Stone-Čech remainder $\beta X \setminus X$ is closed in $\beta X \setminus X$, then $X$ is ACRIN. They asked if each normal space whose Stone-Čech remainder has countable tightness is ACRIN. Theorem 2 gives the positive answer assuming the Proper Forcing Axiom.

It is well known that the tightness of $\beta \omega \setminus \omega$ is $2^\omega$. Since every not countably compact Hausdorff space contains a closed copy of $\omega$, the next lemma is easy to prove.

Lemma 1. If $X$ is a normal space and $\beta X \setminus X$ has countable tightness, then $X$ is countably compact.

Theorem 2 (PFA). If $X$ is a normal space and $\beta X \setminus X$ has countable tightness, then $X$ is ACRIN.

Proof. Let $f: X \to Y$ be a continuous map and $Y$ be regular. We prove that $Y$ is normal. By virtue of Lemma 5 of [FL] there exist $Z$ and $bf$ such that $X \subseteq Z \subseteq \beta X$ and $bf$ is a perfect map from $Z$ onto $Y$ with $bf|_X = f$. Since $\beta X \setminus X$ has countable tightness, it is easy to see that the spaces $X$, $Y$, and $Z$ are all countably compact. Since perfect mappings preserve normality, we only need to prove that $Z$ is normal. Let $K$ and $L$ be two disjoint closed subsets of $Z$. We will prove that $K \cap \beta X \setminus X = \emptyset$. Since $X$ is normal, we have $K \cap X \beta X \cap L \cap X \beta X = \emptyset$. Take open subsets $U$ and $V$ of $\beta X$ such that $U \beta X \cap V \beta X = \emptyset$, $K \cap X \beta X \subseteq U$, and $L \cap X \beta X \subseteq V$. Let $U' = U \setminus L \beta X$, $V' = V \setminus K \beta X$, and $T = (K \cup L \beta X) \setminus (U' \cup V')$. Obviously $T$ is a closed subset of $\beta X$ and is contained in $\beta X \setminus X$. Thus $T$ is a compact space of countable tightness. Furthermore, $K \setminus U$ and $L \setminus V$ are contained in $T$ and are closed subsets of the countably compact space $Z$. We have proved that $K \setminus U$ and $L \setminus V$ are countably compact subsets of a compact space of countable tightness. By virtue of Balogh’s Theorem [Ba, 2.1], $K \setminus U$ and $L \setminus V$ are compact. Thus
we have
\[
\overline{K}^{\beta X} \cap \overline{L}^{\beta X} = (\overline{K \cap U}^{\beta X} \cup (K \setminus U)) \cap (\overline{L \cap V}^{\beta X} \cup (L \setminus V))
\]
\[
= (K \cap U^{\beta X} \cap (L \setminus V)) \cup (L \cap V^{\beta X} \cap (K \setminus U))
\]
\[
\subseteq (K^{\beta X} \cap L) \cup (L^{\beta X} \cap K) = \emptyset.
\]
We are done.

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REFERENCES


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