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THE COMPLETE CONTINUITY PROPERTY IN BOCHNER FUNCTION SPACES

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ABSTRACT. Let X be a Banach space, $(\Omega, \Sigma, \lambda)$ a finite measure space, and $1 < p < \infty$. It is shown that $L^p(\lambda, X)$ has the complete continuity property if and only if X has it. A similar result about $L^1_\Lambda(G, X)$ is also given.

I. INTRODUCTION

Let X be a Banach space, let $(\Omega, \Sigma, \lambda)$ be a finite measure space, and let $1 \leq p < \infty$. We denote by $L^p(\lambda, X)$ the Banach space of all (class of) X-valued p-Bochner λ-integrable functions (class of) with its usual norm. If X is the scalar field then $L^p(\lambda, X)$ will be denoted by $L^p(\lambda)$.

A Banach space X is said to have the complete continuity property if for every finite measure space $(K, \mathscr{F}, \mu)$, every bounded operator $T: L^1(K, \mathscr{F}, \mu) \to X$ is a Dunford-Pettis operator. Any Banach space with the Radon-Nikodým property (RNP) has the complete continuity property. In particular, any $L^p(\lambda)$, $1 < p < \infty$, has the complete continuity property. It is well known (see [DU]) that if X has the (RNP) then $L^p(\lambda, X)$ has the same property. Recently, Saab and Saab [SS] observed that if X is a dual Banach space that has the complete continuity property then $L^p(\lambda, X)$ enjoys the same property. They also asked [SS, Question 13] whether $L^p(\lambda, X)$ has the complete continuity property whenever X does.

In this paper we will show that the answer is always affirmative. The question of when a property passes from the Banach space X to $L^p(\lambda, X)$ was extensively studied by several authors in the past. Let us recall that Kwapien [Kw] showed that $L^p(\lambda, X)$ $(1 \leq p < \infty)$ contains a copy of c_0 if and only if X contains a copy of c_0. Talagrand [T] showed that if X is weakly sequentially complete then $L^p(\lambda, X)$ $(1 \leq p < \infty)$ is weakly sequentially complete. Kalton, Saab, and Saab [KSS] were able to prove that the property (u) also passes from X to $L^p(\lambda, X)$ $(1 \leq p < \infty)$. Mendoza [M] succeeded in showing that X contains a complemented copy of l_1 if and only if $L^p(\lambda, X)$ $(1 < p < \infty)$

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contains a complemented copy of l_1. The notation and terminology used and not defined in this paper can be found in [D, DU].

II. Preliminaries

Definition 1. Let E and F be Banach spaces and suppose $T: E \to F$ is a bounded linear operator. T is said to be a *Dunford-Pettis* operator if whenever $(x_n)_n$ is a weakly null sequence in E, $(Tx_n)_n$ is a norm-null sequence in F.

It is known (see [DU]) that for a Banach space X, an operator $T: L^1(\mu) \to X$ is a Dunford-Pettis operator if and only if its restriction on $L^\infty(\mu)$ is a compact operator.

Definition 2. Let E and F be Banach spaces and suppose $T: E \to F$ is a bounded linear operator. T is said to be an *absolutely summing* operator if there is a constant $C > 0$ such that for any finite set $(x_m)_{1 \le m \le n}$ in E the following inequality holds:

$$\sum_{m=1}^{n} \|Tx_m\| \le C \sup \left\{ \sum_{m=1}^{n} |x^*(x_m)|;\ x^* \in E^*,\ \|x^*\| \le 1 \right\}.$$

The least possible constant C for which the above inequality holds is denoted by $\|T\|_{\mathrm{as}}$. It is easy to check that the class of all absolutely summing operators from E to F is a Banach space under the norm $\|T\|_{\mathrm{as}}$. This Banach space will be denoted by $\Pi_1(E, F)$.

Equivalent formulations for the operator $T: E \to F$ to be absolutely summing can be found in [DU, p. 162]. If E happens to be a $C(K)$ space where K is a compact Hausdorff space, we recall that T is absolutely summing if and only if its representing measure G (see [DU, p. 152]) is of bounded variation, and in this case $\|T\|_{\mathrm{as}} = |G|(K)$ where $|G|(K)$ denotes the total variation of G. In this paper we take advantage of this fact by identifying the two Banach spaces $\Pi_1(C(K), F)$ and $M(K, X)$ as the space of X-valued measures that are of bounded variations.

To be able to prove our result we make crucial use of the following representation theorem of Kalton. Let us denote by $\beta(K)$ the sigma algebra of Borel subsets of K.

Theorem 1 (Kalton [K]). *Suppose that*

(i) *K is a compact metric space and μ is a Radon probability measure on K;*
(ii) *Ω is a Polish space and λ is a Radon probability measure on Ω;*
(iii) *X is a separable Banach space;*
(iv) *$T: L^1(\mu) \to L^1(\lambda, X)$ is a bounded linear operator.*

Then there is a map $\omega \to T_\omega(\Omega \to \Pi_1(C(K), X))$ such that for every $f \in C(K)$, the map $\omega \to T_\omega(f)$ is Borel measurable form Ω to X and

(a) *If μ_ω is the representing measure of T_ω then $\int_\Omega |\mu_\omega|(B)\, d\lambda(\omega) \le \|T\|\mu(B)$ for $B \in \beta(K)$;*
(b) *If $f \in L^1(\mu)$ then λ-a.e. one has $f \in L^1(|\mu_\omega|)$;*
(c) *$Tf(\omega) = T_\omega f$ λ a.e. for every $f \in L^1(\mu)$.*

III. The complete continuity property for spaces of Bochner integrable functions

Theorem 2. *Let X be a Banach space, let $1 < p < \infty$, and let $(\Omega, \Sigma, \lambda)$ be a finite measure space. Then $L^p(\lambda, X)$ has the complete continuity property whenever X does.*

In the sequel we can and do assume, without loss of generality, that X is separable, K and Ω are compact metric spaces with μ and λ two Radon measures on their respective Borel σ-algebras.

We will divide the proof into two propositions.

Proposition 1. *Suppose X is a Banach space having the complete continuity property, and let $T: L^1(\mu) \to L^\infty(\lambda, X)$ be a bounded linear operator. If we denote by $i_p: L^\infty(\lambda, X) \to L^p(\lambda, X)$ the natural inclusion then $i_p \circ T$ is a Dunford Pettis operator for each $1 \le p < \infty$.*

Proof. Fix a strongly Borel measurable map $\omega \to T_\omega(\Omega \to \Pi_1(C(K), X))$ as in Theorem 1. We need

Lemma 1. *For λ-almost every ω, the operator T_ω can be extended as a bounded linear operator $L^1(\mu) \to X$.*

Proof. To see this, let $(f_n)_{n\ge1}$ be a countable dense subset of $C(K)$. Using the fact that for each n, $\|Tf_n\|_\infty \le \|T\|\,\|f_n\|$, one can choose a measurable subset Ω_n of Ω such that $\lambda(\Omega\backslash\Omega_n) = 0$ and $\|T_\omega f_n\| \le \|T\|\,\|f_n\|$ for each $\omega \in \Omega_n$. Taking $\widetilde{\Omega} = \bigcap_n \Omega_n$, we have $\lambda(\Omega\backslash\widetilde{\Omega}) = 0$ and $\|T_\omega f_n\| \le \|T\|\,\|f_n\|$ for each $n \ge 1$ and $\omega \in \widetilde{\Omega}$. Since $\{f_n\}_{n\ge1}$ is also dense in $L^1(\mu)$, T_ω can be extended to a bounded operator on $L^1(\mu)$.

Now let $(g_j)_j$ be a weakly null sequence in the unit ball of $L^1(\mu)$. Since X has the complete continuity property and the operators $(T_\omega)_\omega$ are bounded, one can deduce that $\|T_\omega(g_j)\|$ converges to 0 for each $\omega \in \widetilde{\Omega}$. Also $\|T_\omega(g_j)\|^p \le \|T\|^p$ for λ-a.e. ω; therefore, by the Lebesgue dominated convergence theorem, $\int \|T_\omega(g_j)\|^p\, d\lambda(\omega)$ converges to 0 (i.e., $\|i_p \circ T(g_j)\| \to 0$), which proves that $i_p \circ T$ is a Dunford-Pettis operator.

Proposition 2. *Let X be a Banach space and $1 < p < \infty$. Suppose $T: L^1(\mu) \to L^p(\lambda, X)$ is a bounded linear operator. Then there exists a sequence $(T_n)_n$, where $T_n \in \mathscr{L}(L^1(\mu), L^\infty(\lambda, X))$ for each n and $(T - i_p \circ T_n)|_{L^\infty(\mu)}$ converges to zero in the norm operator topology.*

Proof. As before, let us consider the representation R

$$\omega \to T_\omega(\Omega \to \Pi_1(C(K), X)).$$

Recall that the space $\Pi_1(C(K), X)$ is isometrically isomorphic to the space $M(K, X)$. Let us denote by S the isometry that associates to each operator T in $\Pi_1(C(K), X)$ its representing measure. With this in mind one can consider the representation as the map $S \circ R: \Omega \to M(K, X)$, which satisfies the following property: for every $f \in C(K)$ the map $\omega \to \int f\, d\mu_\omega$ is measurable as a map from Ω to X where $\mu_\omega = S \circ R(\omega)$.

We need two lemmas to complete the proof.

Lemma 2. *For λ-almost every ω in Ω, we have that $|\mu_\omega|$ is absolutely continuous with respect to μ.*

Proof. To see this, notice that we can always assume X to be separable, and for each $x^* \in X^*$, consider $T_{x^*}: L^1(\mu) \to L^p(\lambda)$ given by $T_{x^*}(f)(\omega) = \langle Tf(\omega), x^*\rangle$. The operator T_{x^*} is a bounded operator and is easily checked to be represented by the map $\omega \to x^*\mu_\omega$. Now since $1 < p < \infty$, T_{x^*} is representable and, by [F, Proposition 3], $|x^*\mu_\omega| \ll \mu$ for almost every ω. To conclude the proof, consider a countable and weak*-dense subset $A = \{x_1^*, x_2^*, \dots\}$ in the unit ball of X^*. For a suitable subset $\widetilde{\Omega}$ of Ω with $\lambda(\Omega\backslash\widetilde{\Omega}) = 0$ we have for each $\omega \in \widetilde{\Omega}$, $|x_n^*\mu_\omega| \ll \mu$, $n \geq 1$. We claim that $|\mu_\omega| \ll \mu$ for each $\omega \in \widetilde{\Omega}$. For that, it is enough to show (see [DU, Theorem I-2-1]) that $\mu_\omega(A) = 0$ whenever $\mu(A) = 0$. Assume $\mu(A) = 0$ and $\omega \in \widetilde{\Omega}$; for each x^* in the unit ball of X^*, there exists a sequence $(x_{n_j}^*)_j \subset A$ so that $x_{n_j}^*$ converges to x^* weak*. Since $\mu_\omega(A) \in X$, $x_{n_j}^*(\mu_\omega(A)) \to_j x^*(\mu_\omega(A))$ and, since $x_n^*(\mu_\omega(A)) = 0$ for each $n \geq 1$, we have $x^*(\mu_\omega(A)) = 0$.

Lemma 3. *The integral $\int_\Omega (|\mu_\omega|(K))^p \, d\lambda(\omega)$ is finite.*

Proof. Let $h(\omega) = |\mu_\omega|(K)$. We will be done if we can show that $h \in L^p(\lambda)$. Let q be such that $1/p + 1/q = 1$, fix $f \in C(\Omega)$ with $\|f\|_q \leq 1$, and consider the map $Tf: L^1(\mu) \to L^1(\lambda, X)$ given by $T_f(g)(\omega) = f(\omega)(Tg)(\omega)$. It is easy to see that T_f is represented by the map $\omega \to f(\omega)\mu_\omega$, and by Theorem 1(a) and Hölder's inequality we have

$$(\dagger) \qquad \int_\Omega |f(\omega)|\, h(\omega)\, d\lambda(\omega) \leq \|T_f\| \leq \|T\|.$$

Now notice that

$$\begin{aligned}\left(\int_\Omega |h(\omega)|^p \, d\lambda(\omega)\right)^{1/p} &= \sup\left\{\int_\Omega s(w)|h(\omega)|\, d\lambda(\omega);\ \|s\|_q \leq 1\right\} \\ &= \sup\left\{\int_\Omega s(w)|h(\omega)|\, d\lambda(\omega);\ \|s\|_q \leq 1,\ s \in C(\Omega)\right\} \leq \|T\|.\end{aligned}$$

The last inequality is derived from $(\dagger)$. Hence $h \in L^p(\lambda)$.

Now we will proceed to the construction of the sequence $(T_n)_n$.

By Lemma 2, for almost every $\omega \in \Omega$ there exists g_w in $L^1(\mu)$, which is the Radon-Nikodým density of $|\mu_\omega|$ with respect to μ. By Theorem 1(a), the map $\omega \to g_\omega(\Omega \to L^1(\mu))$ is norm measurable. Therefore $(\omega, t) \to g_\omega(t)(\Omega \times K \to \mathbb{R})$ is measurable. Let V_n be the measurable subset of $\Omega \times K$ given by $V_n = \{(\omega, t) \mid g_\omega(t) \leq n\}$. The map T_n from $L^1(\mu)$ to $L^\infty(\lambda, X)$ is given by

$$T_n f(\omega) = \int f(t) 1_{V_n}(\omega, t)\, d\mu_\omega(t)$$

for almost every ω. The operator T_n is obviously linear and, for each f in

$L^1(\mu)$, we have

$$\begin{aligned}\|T_n f(\omega)\| &\leq \int |f(t)| 1_{V_n}(\omega, t)\, d|\mu_\omega|(t).\\ &= \int |f(t)| 1_{V_n}(\omega, t) g_\omega(t)\, d\mu(t)\\ &\leq n \int |f(t)|\, d\mu(t) \quad \text{for } \lambda\text{-a.e. } \omega \text{ in } \Omega.\end{aligned}$$

Consequently, $\|T_n\| \leq n$ for each $n \geq 1$. Moreover, if f is in the unit ball of $L^\infty(\mu)$,

$$\begin{aligned}\|(i_p \circ T_n - T)(f)\|^p &\leq \int_\Omega \left\| \int_K f(t) 1_{V_n^c}(\omega, t)\, d\mu_\omega(t) \right\|^p d\lambda(\omega)\\ &\leq \int_\Omega \left(\int_K 1_{V_n^c}(\omega, t) g_\omega(t)\, d\mu(t) \right)^p d\lambda(\omega).\end{aligned}$$

So

$$\|(i_p \circ T_n - T)|_{L^\infty(\mu)}\|^p \leq \int_\Omega \left(\int_K 1_{V_n^c}(\omega, t) g_\omega(t)\, d\mu(t) \right)^p d\lambda(\omega).$$

We claim that for λ-a.e. ω, $\int_K 1_{V_n^c}(\omega, t) g_\omega(t)\, d\mu(t) \to 0$ as $n \to \infty$. To see this, notice that $(\omega, t) \to g_\omega(t)$ is a member of $L^1(\Omega \times K, \mu \otimes \lambda)$ and thus

$$\lim_{n\to\infty} 1_{V_n^c}(\omega, t) g_\omega(t) = 0 \quad \mu \otimes \lambda\text{-a.e.}$$

Therefore, $\int_K 1_{V_n^c}(\omega, t) g_\omega(t)\, d\mu(t) \to 0$ as $n \to \infty$. We also have

$$\int_K 1_{V_n^c}(\omega, t) g_\omega(t)\, d\mu(t) \leq |\mu_\omega|(K)$$

for almost every ω. Now using Lemma 3 and the Lebesgue dominated convergence theorem, we get that

$$\int_\Omega \left(\int_K 1_{V_n^c}(\omega, t) g_\omega(t)\, d\mu(t) \right)^p d\lambda(\omega)$$

converges to 0, which proves that $\|(i_p \circ T_n - T)|_{L^\infty(\mu)}\|$ converges to 0. Now combining Propositions 1 and 2, we get that $T|_{L^\infty(\mu)}$ is a compact operator and, by the remark after Definition 1, T is a Dunford-Pettis operator. The theorem is proved.

For the next theorem, we need to fix some notation and definitions. Let G be a compact metrizable abelian group. Let $\Gamma = \widehat{G}$ be the dual group, the set of continuous homomorphisms $\gamma\colon G \to \mathbf{C}$. We will write λ for the normalized Haar measure on G.

Definition 3. (i) If $f \in L^1(\lambda, X)$ then we denote by $\hat{f}$ the Fourier transform of f that is a map from Γ to X defined by $\hat{f}(\gamma) = \int_G \bar{\gamma} f\, d\lambda$.

(ii) If $\mu \in M(G, X)$ then we denote by $\hat{\mu}$ the Fourier transform of μ that is a map from Γ to X defined by $\hat{\mu}(\gamma) = \int_G \bar{\gamma}\, d\mu$.

If $\Lambda \subseteq \Gamma$ is a set of characters, let

$$\begin{aligned}L^1_\Lambda(G; X) &= \{f \in L^1(\lambda, X) : \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\},\\ M_\Lambda(G, X) &= \{\mu \in M(G, X) : \hat{\mu}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}.\end{aligned}$$

Definition 4. A subset Λ of Γ is called a Riesz set if and only if $M_\Lambda(G, \mathbf{C}) = L^1_\Lambda(G, \mathbf{C})$. For example, the F&M Riesz theorem states that $\mathbb{N}$ is a Riesz set in $\mathbb{Z} = \widehat{\mathbb{T}}$.

Theorem 3. *Let G be a compact metrizable abelian group and let Λ be a Riesz set in Γ. Then if X is a Banach space having the complete continuity property, the space $L^1_\Lambda(G, X)$ has the complete continuity property. In particular, $H^1(\mathbb{T}, X)$ has the complete continuity property whenever X does.*

Proof. The proof is similar to the proof of Theorem 2. What we need to adjust is the proof of Lemma 2, which turns out to be true using the fact that if Λ is a Riesz set then $L^1_\Lambda(G)$ has the (RNP) [L-P].

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