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FEKETE-SZEGÖ INEQUALITIES FOR CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. Let $K(\beta)$ denote the class of normalised close-to-convex functions of order β defined in the unit disc, and let $f \in K(\beta)$ with $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Sharp bounds are obtained for $|a_3 - \mu a_2^2|$, where μ is real.

1. INTRODUCTION

Let the function f be given by

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots \qquad (|z| < 1). \tag{1}$$

A classical result of Fekete and Szegö [1] determines the maximum value of $|a_3 - \mu a_2^2|$, as a function of the real parameter μ, for the class of univalent functions f. There are now several results of this type in the literature, each of them dealing with $|a_3 - \mu a_2^2|$ for various classes of functions f (see, e.g., [3–6]).

In this paper we consider the problem for the class $K(\beta)$ of close-to-convex functions of order β in the sense of Pommerenke [7]. Thus $f \in K(\beta)$ if and only if f is given by (1) and for some starlike function g satisfies

$$\left|\arg \frac{zf'(z)}{g(z)}\right| \le \frac{\pi\beta}{2} \tag{2}$$

for $|z| < 1$ and $\beta \ge 0$. A recent paper by Abdel-Gawad and Thomas [2] contains a partial proof of the following theorem, the last two inequalities remaining unproved for $\beta > 1$.

Theorem. *Let $f \in K(\beta)$ and be given by* (1). *Then for* $\beta \ge 0$

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{1}{3}(1 + (2 - 3\mu)(\beta + 1)^2) & \text{if } \mu \le \frac{2\beta}{3(\beta+1)}, \\ \frac{1}{3}\left(1 + 2\beta + \frac{2(2-3\mu)}{2-\beta(2-3\mu)}\right) & \text{if } \frac{2\beta}{3(\beta+1)} \le \mu \le \frac{2}{3}, \\ \frac{1}{3}(1 + 2\beta) & \text{if } \frac{2}{3} \le \mu \le \frac{2(\beta+2)}{3(\beta+1)}, \\ \frac{1}{3}(-1 + (3\mu - 2)(\beta + 1)^2) & \text{if } \mu \ge \frac{2(\beta+2)}{3(\beta+1)}. \end{cases}$$

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For each μ there are functions in $K(\beta)$ such that equality holds in all cases.

We give a simple proof of the complete theorem.

2. Proof of the theorem

We shall require the following:

Lemma [8, pp. 41, 166]. *Let* $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *satisfy* $\operatorname{Re} h(z) > 0$ $(|z| < 1)$. *Then* $|c_k| \le 2$ $(k \ge 1)$ *and*

$$|c_2 - \tfrac{1}{2}c_1^2| \le 2 - \tfrac{1}{2}|c_1|^2.$$

Let $f \in K(\beta)$. Then it follows from (2) that we may write

$$zf'(z) = g(z)h^\beta(z),$$

where g is starlike and h has positive real part. Let $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$, and let h be given as in the lemma above. Then by equating coefficients we obtain

$$2a_2 = \beta c_1 + b_2$$

and

$$3a_3 = \tfrac{1}{2}\beta(\beta - 1)c_1^2 + \beta c_2 + \beta c_1 b_2 + b_3,$$

so, with $X = \frac{1}{2}(2 - 3\mu)$,

$$3(a_3 - \mu a_2^2) = b_3 - \tfrac{3}{4}\mu b_2^2 + \beta(c_2 + \tfrac{1}{2}(\beta X - 1)c_1^2) + \beta X c_1 b_2. \tag{3}$$

Since rotations of f also belong to $K(\beta)$, we may assume, without loss of generality, that $a_3 - \mu a_2^2$ is positive. Thus we now estimate $\operatorname{Re}(a_3 - \mu a_2^2)$.

For some function $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ $(|z| < 1)$ with positive real part, we have $zg'(z) = g(z)p(z)$; hence, by equating coefficients, $b_2 = p_1$ and $b_3 = \frac{1}{2}(p_2 + p_1^2)$. So by the lemma,

$$\begin{aligned}\operatorname{Re}(b_3 - \tfrac{3}{4}\mu b_2^2) &= \tfrac{1}{2}\operatorname{Re}(p_2 - \tfrac{1}{2}p_1^2) + \tfrac{3}{4}(1 - \mu)\operatorname{Re} p_1^2 \\ &\le 1 - \rho^2 + (1 + 2X)\rho^2 \cos 2\phi,\end{aligned} \tag{4}$$

where $b_2 = p_1 = 2\rho e^{i\phi}$ for some ρ in $[0, 1]$. We also have

$$\begin{aligned}\operatorname{Re}(c_2 + \tfrac{1}{2}(\beta X - 1)c_1^2) &= \operatorname{Re}(c_2 - \tfrac{1}{2}c_1^2) + \tfrac{1}{2}\beta X \operatorname{Re} c_1^2 \\ &\le 2(1 - r^2) + 2\beta X r^2 \cos 2\theta,\end{aligned} \tag{5}$$

where $c_1 = 2re^{i\theta}$ for some r in $[0, 1]$. From (3)–(5) we obtain

$$\begin{aligned}\operatorname{Re} 3(a_3 - \mu a_2^2) \le{}& 1 - \rho^2 + (1 + 2X)\rho^2 \cos 2\phi + 2\beta(1 - r^2 + r^2 \beta X \cos 2\theta) \\ &+ 4\beta X r\rho \cos(\theta + \phi),\end{aligned} \tag{6}$$

and we now proceed to maximize the right-hand side of (6). This function will be denoted $\psi(X)$ whenever all the parameters except X are held constant.

Assume that $2\beta/3(\beta + 1) \le \mu \le 2/3$ so that $0 \le X \le 1(1 + \beta)$. The expression $-t^2 + t^2 \beta X \cos 2\theta + 2Xt$ is largest when $t = X/(1 - \beta X \cos 2\theta)$, so

$$-t^2 + t^2 \beta X \cos 2\theta + 2Xt \le \frac{X^2}{1 - \beta X \cos 2\theta} \le \frac{X^2}{1 - \beta X}.$$

Thus

$$\psi(X) \le 1 + 2X + 2\beta\left(1 + \frac{X^2}{1-\beta X}\right) = 1 + 2\beta + \frac{2(2-3\mu)}{2-\beta(2-3\mu)},$$

and with (6) this establishes the second inequality in the theorem.

It is now to prove the first inequality. Let $\mu < 2\beta/3(\beta+1)$, so that $X > 1/(1+\beta)$. With $X_0 = 1/(1+\beta)$ we have

$$\begin{aligned}\psi(X) &= \psi(X_0) + 2(X - X_0)(\rho^2\cos 2\phi + \beta^2 r^2 \cos 2\theta + 2\rho\beta r\cos(\theta+\phi))\\ &\le \psi(X_0) + 2(X-X_0)(\beta+1)^2 \le 1 + (2-3\mu)(\beta+1)^2\end{aligned}$$

as required.

Let $X_1 = -1/(1+\beta)$. We shall find that $\psi(X_1) \le 2\beta+1$, and the remaining inequalities follow easily from this one. By an argument similar to the one above, we obtain

$$\psi(X) \le \psi(X_1) + 2|X - X_1|(\beta+1)^2 \le -1 + (3\mu-2)(\beta+1)^2$$

if $X \le X_1$, that is, $\mu \ge 2(\beta+2)/3(\beta+1)$. Also, for $0 \le \lambda \le 1$,

$$\begin{aligned}\psi(\lambda X_1) &= \lambda\psi(X_1) + (1-\lambda)\psi(0)\\ &\le \lambda(2\beta+1) + (1-\lambda)(2\beta+1) = 2\beta+1,\end{aligned}$$

so $\psi(X) \le 2\beta+1$ for $X_1 \le X \le 0$, i.e., $2/3 \le \mu \le 2(\beta+2)/3(\beta+1)$.

We now show that $\psi(X_1) \le 2\beta+1$. We have

$$-t^2 + t^2\beta X\cos 2\theta + 2Xt\rho\cos(\theta+\phi) \le \frac{X^2\rho^2\cos 2(\theta+\phi)}{1-\beta X\cos 2\theta}$$

for all real t, so

$$\psi(X) - 1 - 2\beta \le \rho^2\left[-1 + (1+2X)\cos 2\phi + \frac{\beta X^2(1+\cos 2(\theta+\phi))}{1-\beta X\cos 2\theta}\right].$$

Thus we consider the inequality

$$\beta X^2(1+\cos 2(\theta+\phi)) + (1-\beta X\cos 2\theta)(-1 + (1+2X)\cos 2\phi) \le 0$$

with $X = X_1$. After some simplification this becomes

$$\beta^2(\cos 2\phi - 1)(\cos 2\theta + 1) - \beta(1 + \cos 2\theta + \sin 2\theta\sin 2\phi) - 1 - \cos 2\phi \le 0,$$

which is true if

$$2\beta^2\sin^2\phi\cos^2\theta + 2\beta\cos\theta\sin\theta\cos\phi\sin\phi + \cos^2\phi \ge 0. \tag{7}$$

Now, for all real t,

$$2t^2 + 2\sin\theta\cos\phi t + \cos^2\phi \ge 0,$$

so by taking $t = \beta\sin\phi\cos\theta$, we obtain (7). This completes the proof of the inequalities.

An examination of the proof shows that the four inequalities in the theorem are sharp if we take $c_1 = c_2 = b_2 = 2$, $b_3 = 3$ in the first case; $c_1 = 2(2-3\mu)/2 - \beta(2-3\mu)$, $c_2 = b_2 = 2$, $b_3 = 3$ in the second; $c_1 = b_2 = 0$,

$c_2 = 2$, $b_3 = 1$ in the third; and $c_1 = b_2 = 2i$, $c_2 = -2$, $b_3 = -3$ in the last. The corresponding functions f may be defined by taking $f'(z)$ respectively as

$$\frac{1}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^\beta, \qquad \frac{1}{(1-z)^2}\left(\lambda\frac{1+z}{1-z} + (1-\lambda)\frac{1-z}{1+z}\right)^\beta,$$
$$\frac{1}{1-z^2}\left(\frac{1+z^2}{1-z^2}\right)^\beta, \qquad \frac{1}{(1-iz)^2}\left(\frac{1+iz}{1-iz}\right)^\beta,$$

where, in the second case,

$$\lambda = \frac{2 + (1-\beta)(2-3\mu)}{2(2-\beta(2-3\mu))}.$$

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