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INTEGRABILITY OF SUPERHARMONIC FUNCTIONS ON HÖLDER DOMAINS OF THE PLANE

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ABSTRACT. We prove that if D is a finitely connected Hölder domain of the plane, then there exists $p > 0$ for which every positive superharmonic function on D is p-integrable over D with respect to two-dimensional Lebesgue measure.

1. INTRODUCTION

Let D be a proper subdomain of the complex plane $\mathbb{C}$. We denote by $\delta_D(z)$ the distance between a point $z \in D$ and the boundary ∂D of D:

$$\delta_D(z) = \inf\{|z - \zeta| \,|\, \zeta \in \partial D\}.$$

The quasi-hyperbolic metric k_D in D is defined by

$$k_D(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{\delta_D(z)}, \qquad z_1, z_2 \in D,$$

where the infimum is taken over all rectifiable arcs γ joining z_1 and z_2 in D (cf. Gehring-Palka [4] and Gehring-Osgood [3]). Let $a \in D$. If there are constants α and A such that

$$(1) \qquad k_D(a, z) \le \alpha \log \frac{\delta_D(a)}{\delta_D(z)} + A, \qquad z \in D,$$

then D is called a Hölder domain.

Denote by $S^+(D)$ (resp. $H^+(D)$) the set of positive superharmonic (resp. harmonic) functions on D and by $L^p(D)$, $0 < p < +\infty$, the spaces of Lebesgue measurable functions on D which are p-integrable over D with respect to two-dimensional Lebesgue measure. We ask whether there is $p > 0$ such that $S^+(D) \subset L^p(D)$ (or $H^+(D) \subset L^p(D)$). It is clear that if for each $z_0 \in D$ there is $M > 0$ such that the inequality

$$(2) \qquad \iint_D u(z)^p \, dx\, dy \le M u(z_0)^p, \qquad z = x + iy,$$

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holds for all $u \in S^+(D)$ (resp. $u \in H^+(D)$), then $S^+(D) \subset L^p(D)$ (resp. $H^+(D) \subset L^p(D)$). Conversely, if $S^+(D) \subset L^p(D)$ (resp. $H^+(D) \subset L^p(D)$), then for each $z_0 \in D$ there is $M > 0$ such that (2) holds for all $u \in S^+(D)$ (resp. $u \in H^+(D)$) (see Suzuki [13]).

In 1972 Armitage [1] proved that $S^+(D) \subset L^p(D)$ for $0 < p < 2$ if D has bounded curvature. This result was generalized by Maeda and Suzuki [7] to the case of Lipschitz domains. In our previous work [9] we considered domains satisfying an interior wedge condition and improved their results. Recently, Smith and Stegenga have proved the following theorem:

Theorem A (Smith-Stegenga [11, Corollary 4]). *Let D be a finitely connected domain of $\mathbb{C}$. Then the following are equivalent*:

(i) *D is a Hölder domain.*
(ii) *$H^+(D) \subset L^p(D)$ for some $p > 0$.*

The purpose of this paper is to prove the following

Theorem. *If D is a finitely connected Hölder domain of $\mathbb{C}$, then $S^+(D) \subset L^p(D)$ for some $p > 0$.*

Combining this theorem with Theorem A, we obtain the following result:

Corollary. *Let D be a finitely connected domain of $\mathbb{C}$. Then the following are equivalent*:

(i) *D is a Hölder domain.*
(ii) *$H^+(D) \subset L^p(D)$ for some $p > 0$.*
(iii) *$S^+(D) \subset L^p(D)$ for some $p > 0$.*

It is trivial that if $S^+(D) \subset L^p(D)$ for some $p > 0$, then $H^+(D) \subset L^p(D)$ for the *same* p. However, it is not known whether the converse is true. By the Riesz decomposition theorem (cf. Helms [5, Theorem 6.18]) every positive superharmonic function on D is uniquely decomposed into the sum of a (Green) potential and a positive harmonic function on D. It should be noted that if the set of potentials on D is embedded into $L^p(D)$ for some $p > 0$, then $S^+(D) \subset L^p(D)$ for the *same* p. This observation is due to Armitage [1] and Suzuki [13, Corollary 1] and will be implicitly used in the proof of our theorem.

To prove the theorem we employ a method similar to that used in [9]. In the next section we will prepare a lemma concerning the boundary behavior of conformal mappings. Using this lemma, we will prove the theorem in §3.

2. Distortion of conformal mappings

Let D be a domain of the Riemann sphere $\widehat{\mathbb{C}}$ whose complement $\widehat{\mathbb{C}} \setminus D$ consists of more than two points. The universal covering surface of D is conformally equivalent to the unit disk Δ. The Poincaré metric $\lambda_D(z)|dz|$ for D is defined by

$$\lambda_D(\pi(\zeta))|\pi'(\zeta)| = 1/(1 - |\zeta|^2), \qquad \zeta \in \Delta,$$

where $\pi\colon \Delta \to D$ is a holomorphic universal covering map.

The following properties of the Poincaré metric are well known.

Lemma 1. (i) *If f is a conformal mapping, then $\lambda_{f(D)}(f(z))|f'(z)| = \lambda_D(z)$ for $z \in D$.*

(ii) *If $D_1 \subset D_2$, then $\lambda_{D_2}(z) \le \lambda_{D_1}(z)$ for $z \in D_1$.*

(iii) *The inequality $\lambda_D(z)\delta_D(z) \le 1$ holds for $z \in D$.*

(iv) *If D is a finitely connected domain such that every component of $\widehat{\mathbb{C}}\setminus D$ contains at least two points and $\infty \notin D$, then there exists $M > 0$ for which $\lambda_D(z)\delta_D(z) \ge M$ for all $z \in D$.*

(v) *If D is simply connected and contains ∞, then for each compact set K of $\mathbb{C}$ there exists $M > 0$ such that $\lambda_D(z)\delta_D(z) \ge M$ for all $z \in D \cap K$.*

For a proof of Lemma 1, see, for example, Kra [6, Chapter II, Proposition 1.1] and Masumoto [8, Lemmas 2 and 3].

We denote by h_D the hyperbolic metric on D:

$$h_D(z_1, z_2) = \inf_\gamma \int_\gamma \lambda_D(z)\,|dz|, \qquad z_1, z_2 \in D,$$

where the infimum is taken over all rectifiable arcs γ joining z_1 and z_2 in D. By Lemma 1(iii) and (iv), if D is a finitely connected domain such that every component of $\widehat{\mathbb{C}}\setminus D$ is a continuum and $\infty \notin D$, then there is $M > 0$ for which

$$(3) \qquad Mk_D(z_1, z_2) \le h_D(z_1, z_2) \le k_D(z_1, z_2), \qquad z_1, z_2 \in D.$$

A bounded domain of $\mathbb{C}$ is called regular if it is bounded by finitely many simple closed analytic curves. Modifying the arguments in Becker-Pommerenke [2], we can establish the following

Lemma 2. *If f is a conformal mapping of a regular domain D_0 of $\mathbb{C}$ onto a Hölder domain D, then there exist $M > 0$ and $0 < \beta < 1$ such that*

$$|f'(z)| \le M\lambda_{D_0}(z)^\beta$$

for all $z \in D_0$.

Proof. We assume that there are constants α and A for which (1) holds. Since

$$(4) \qquad k_D(a, z) \ge \log(\delta_D(a)/\delta_D(z))$$

by Gehring-Palka [4, Lemma 2.1], we have $\alpha \ge 1$. It is thus sufficient to show that $|f'(z)|\lambda_{D_0}(z)^{-1+1/2\alpha}$ is bounded on D_0.

The conformal mapping f induces a bijection between the set of components of ∂D_0 and the set of components of $\widehat{\mathbb{C}}\setminus D$. Let C_0 be a component of ∂D_0 and let C be the component of $\widehat{\mathbb{C}}\setminus D$ corresponding to C_0 under the bijection. Set $D_1 = \widehat{\mathbb{C}}\setminus C$.

Assume first that $\infty \in D_1$. Let f_1 be a conformal mapping of $\Delta^* = \widehat{\mathbb{C}}\setminus\overline{\Delta}$, the outside of the unit disk Δ, onto D_1 with $f_1(\infty) = \infty$. Then, in view of

Lemma 1 and (1) and (3), we see that, for $\zeta \in \Delta^*$ sufficiently near $\partial\Delta^*$,

$$\begin{aligned}\log \lambda_{\Delta^*}(\zeta) &= \log \frac{1}{|\zeta|^2 - 1} \le \log \frac{|\zeta| + 1}{|\zeta| - 1} = 2h_{D_1}(f_1(\zeta), \infty) \\ &\le 2\{h_{D_1}(f_1(\zeta), a) + h_{D_1}(a, \infty)\} \le 2h_D(f_1(\zeta), a) + 2h_{D_1}(a, \infty) \\ &\le 2k_D(f_1(\zeta), a) + 2h_{D_1}(a, \infty) \\ &\le 2\alpha \log \frac{\delta_D(a)}{\delta_D(f_1(\zeta))} + 2A + 2h_{D_1}(a, \infty) \\ &= 2\alpha \log \frac{1}{\delta_{D_1}(f_1(\zeta))} + M_1 \le 2\alpha \log\{M_2 \lambda_{D_1}(f_1(\zeta))\} + M_1 \\ &= 2\alpha \log \frac{\lambda_{\Delta^*}(\zeta)}{|f_1'(\zeta)|} + (2\alpha \log M_2 + M_1),\end{aligned}$$

where M_1 and M_2 are constants. Thus there exists $M_3 > 0$ such that

$$|f_1'(\zeta)|\lambda_{\Delta^*}(\zeta)^{-1+1/2\alpha} \le M_3$$

for $\zeta \in \Delta^*$ near $\partial\Delta^*$. Set $\varphi = f_1^{-1} \circ f$ and $\Delta_0^* = f_1^{-1}(D) \subset \Delta^*$. Since φ is extended to a homeomorphism of $D_0 \cup C_0$ onto $\Delta_0^* \cup \partial\Delta^*$ and both C_0 and $\partial\Delta^*$ are analytic, we can continue φ conformally beyond C_0 by the reflection principle. In particular, $|\varphi'(z)| \le M_4$ in a neighborhood of C_0. Hence,

$$\begin{aligned}|f'(z)|\lambda_{D_0}(z)^{-1+1/2\alpha} &= |f_1'(\varphi(z))|\,|\varphi'(z)|\lambda_{D_0}(z)^{-1+1/2\alpha} \\ &\le M_3 \lambda_{\Delta^*}(\varphi(z))^{1-1/2\alpha} |\varphi'(z)| \lambda_{D_0}(z)^{-1+1/2\alpha} \\ &\le M_3 |\varphi'(z)| \left(\frac{\lambda_{\Delta_0^*}(\varphi(z))}{\lambda_{D_0}(z)}\right)^{1-1/2\alpha} \\ &= M_3 |\varphi'(z)|^{1/2\alpha} \le M_3 M_4^{1/2\alpha}\end{aligned}$$

for z near C_0.

The case $\infty \notin D_1$ can be treated similarly. Let f_2 be a conformal mapping of Δ onto D_1 with $f_2(0) = a$. Then, for $z \in \Delta$ sufficiently near $\partial\Delta$,

$$\begin{aligned}\log \lambda_\Delta(\zeta) &\le 2h_{D_1}(f_2(\zeta), a) \le 2k_D(f_2(\zeta), a) \\ &\le 2\alpha \log \frac{\delta_D(a)}{\delta_D(f_2(\zeta))} + 2A \le 2\alpha \log \frac{\lambda_\Delta(\zeta)}{|f_2'(\zeta)|} + M_5,\end{aligned}$$

which implies that

$$|f_2'(\zeta)|\lambda_\Delta(\zeta)^{-1+1/2\alpha} \le M_6.$$

Therefore, as in the preceding case, we see that $|f'(z)|\lambda_{D_0}(z)^{-1+1/2\alpha}$ is bounded near C_0.

We have shown that $|f'(z)|\lambda_{D_0}(z)^{-1+1/2\alpha}$ is bounded near ∂D_0. Since it is continuous in D_0, it is bounded in D_0, as desired.

3. Proof of the theorem

In this section we give a proof of our theorem stated in the introduction. Let D be a finitely connected Hölder domain of $\mathbb{C}$. Note that D is bounded by Smith-Stegenga [10, Corollary 1(a)]. If $\mathbb{C} \backslash D$ has an isolated point, say ω, then $D \cup \{\omega\}$ is also a Hölder domain. Furthermore, every positive superharmonic function on D is extended to a positive superharmonic function on $D \cup \{\omega\}$. We may thus assume that each component of $\mathbb{C}\backslash D$ is a continuum.

Fix $z_0 \in D$ and consider the set E of points $(p, \gamma) \in [0, +\infty) \times \mathbb{R}$ such that there exists $M > 0$ for which

$$\iint_D \lambda_D(z)^{-\gamma} u(z)^p \, dx\, dy \le M u(z_0)^p \tag{5}$$

for all $u \in S^+(D)$. (We define $+\infty^0 = +\infty$.)

There is a regular domain D_0 of $\mathbb{C}$ which is conformally equivalent to D; let $f\colon D_0 \to D$ be conformal. By Lemma 2 we can choose constants $0 < \beta < 1$ and $M_1 > 0$ such that

$$|f'(\zeta)| \le M_1 \lambda_{D_0}(\zeta)^\beta$$

for $\zeta \in D_0$. Let $G_D(\cdot, w)$ denote the Green's function for D with pole at w. If $\gamma > (2\beta - 1)/(1-\beta)$, then $\gamma - \beta(\gamma+2) > -1$; hence there is $M_2 > 0$ such that

$$\begin{aligned}
&\iint_D \lambda_D(z)^{-\gamma} G_D(z, w)\, dx\, dy \\
&\quad = \iint_{D_0} \lambda_{D_0}(\zeta)^{-\gamma} G_{D_0}(\zeta, f^{-1}(w)) |f'(\zeta)|^{\gamma+2}\, d\xi\, d\eta \\
&\quad \le M_1^{\gamma+2} \iint_{D_0} \lambda_{D_0}(\zeta)^{-\gamma+\beta(\gamma+2)} G_{D_0}(\zeta, f^{-1}(w))\, d\xi\, d\eta \\
&\quad \le M_2 G_{D_0}(f^{-1}(z_0), f^{-1}(w)) = M_2 G_D(z_0, w)
\end{aligned}$$

for all $w \in D$ (cf. [9, Lemma 4]). Thus, for the (Green) potential u of a positive measure μ on D, we have

$$\begin{aligned}
\iint_D \lambda_D(z)^{-\gamma} u(z)\, dx\, dy &= \iint_D \left(\int_D G_D(z, w)\, d\mu(w) \right) \lambda_D(z)^{-\gamma}\, dx\, dy \\
&= \int_D \left(\iint_D G_D(z, w) \lambda_D(z)^{-\gamma}\, dx\, dy \right) d\mu(w) \\
&\le \int_D M_2 G_D(z_0, w)\, d\mu(w) = M_2 u(z_0).
\end{aligned}$$

We have shown that (5) holds for all potentials u on D with $M = M_2$ and $p = 1$. Since every positive superharmonic function on D is approximated by an increasing sequence of potentials on D, the same inequality holds for all $u \in S^+(D)$. In other words, E includes the half line

$$E_1 = \{1\} \times \left(\frac{2\beta - 1}{1 - \beta}, +\infty \right).$$

Next, by Smith-Stegenga [12, Theorem A], there is $\tau > 0$ such that

$$\iint_D \exp(\tau k_D(a, z))\, dx\, dy < +\infty. \tag{6}$$

Combining (6) with (4), we obtain

$$\iint_D \delta_D(z)^{-\tau}\, dx\, dy < +\infty$$

or, equivalently,

$$\iint_D \lambda_D(z)^\tau \, dx\, dy < +\infty ,$$

which implies that E includes the half line

$$E_2 = \{0\} \times [-\tau, +\infty) .$$

Now, since E is a convex set by Hölder's inequality, it includes the convex hull of $E_1 \cup E_2$. Consequently, there exists $p > 0$, independently of z_0, for which the point $(p, 0)$ is contained in E. This means that $S^+(D) \subset L^p(D)$, and the proof is complete.

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