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REPRESENTATIONS OF AlgLat(T)

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Dedicated to George Maltese on his 60th birthday

ABSTRACT. For a hyponormal operator T with the property that the boundary of the essential spectrum is of planar Lebesgue measure zero, it is proved that the operator algebra AlgLat(T) generated by the invariant subspace lattice of T is commutative. If in addition T is a pure hyponormal operator, then AlgLat(T) is shown to be contained in the bicommutant of T. These are particular cases of more general results obtained for restrictions and quotients of operators decomposable in the sense of Foiaş.

An operator $T \in L(H)$ on a complex Hilbert space is called reflexive if the operator algebra AlgLat(T) generated by the invariant subspace lattice of T is as small as it can be, namely, coincides with the closure of the algebra of all polynomials in T with respect to the weak operator topology. In [16] Sarason proved that normal operators and analytic Toeplitz operators are reflexive. In [7] Deddens was able to show that all isometries are reflexive. Using the Scott Brown technique Olin and Thomson [15] proved that, more general, all subnormal operators are reflexive. In 1987 Scott Brown [3] applied his methods to prove invariant subspace results for hyponormal operators. In [4] Chevreau, Exner, and Pearcy formulated the conjecture that all hyponormal operators are reflexive.

As a modest step in this direction we shall show that for each hyponormal operator $T \in L(H)$, for which the boundary of the essential spectrum has planar Lebesgue measure zero, the algebra AlgLat(T) is commutative. If, in addition, T is pure, i.e., has no nontrivial normal reducing parts, then AlgLat(T) is shown to be contained in the bicommutant of T.

1. FREDHOLM THEORY

Let us denote by Y and Z complex Banach spaces that are dual to each other in the sense that either $Z = Y'$ or $Y = Z'$. We fix continuous linear operators $A \in L(Y)$, $B \in L(Z)$ with

$$\langle Ay, z\rangle = \langle y, Bz\rangle \qquad (y \in Y, z \in Z).$$

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Our aim is to represent the operator algebra generated by the invariant subspace lattice of A as an algebra of bounded analytic functions on a suitable open subset of the complex plane.

As usual we denote by $\sigma_e(A)$ the essential spectrum of A, i.e., the set of all complex numbers λ such that $\mathrm{Ker}(\lambda - A)$ or $Y/\mathrm{Im}(\lambda - A)$ is infinite dimensional. Moreover, we write $\mathrm{Lat}(A)$ for the lattice of all closed invariant subspaces of A. Let us consider the operator algebra σ-AlgLat(A) consisting of all $\sigma(Y, Z)$-continuous linear operators in $L(Y)$ that leave invariant all $\sigma(Y, Z)$-closed spaces in $\mathrm{Lat}(A)$. If H is a hole in $\sigma_e(A)$, i.e., a bounded component of $\rho_e(A) = \mathbb{C}\backslash\sigma_e(A)$, then we write $\mathrm{ind}_H(A)$ for the constant value of the index $\mathrm{ind}(\lambda - A)$ on H. There is an elementary way to represent σ-AlgLat(A) as an algebra of bounded analytic functions on the holes H of $\sigma_e(A)$, which are contained in the spectrum $\sigma(A)$ of A.

Let us fix an operator $C \in \sigma$-AlgLat(A) and a hole $H \subset \sigma(A)$ of $\sigma_e(A)$. If $\mathrm{ind}_H(A) \leq 0$, then by a result of Finch [9, Theorem 9] for each $\lambda_0 \in H$ there is an open neighbourhood U of λ_0 and an analytic function $f \in O(U, Z)$ without zeros on U such that $(\lambda - B)f(\lambda) = 0$ $(\lambda \in U)$. If $\mathrm{ind}_H(A) > 0$, then everything remains true with Z and B replaced by Y and A. Since $C' \in \sigma$-AlgLat(B), in the first case for each $\lambda \in H$ there is a unique complex number denoted by $h(\lambda)$ with

$$C'_{|\mathrm{Ker}(\lambda - B)} = h(\lambda) I_{\mathrm{Ker}(\lambda - B)}.$$

In the second case $h(\lambda)$ is defined by the relation

$$C_{|\mathrm{Ker}(\lambda - A)} = h(\lambda) I_{\mathrm{Ker}(\lambda - A)}.$$

Lemma 1.1. *In both cases we obtain a norm continuous algebra homomorphism*

$$\Phi_H \colon \sigma\text{-AlgLat}(A) \to H^\infty(H), \qquad C \to h$$

with $\Phi(p(A)) = p$ *for all complex polynomials* p.

Proof. We only show why for a given operator C the function h is analytic. If $\mathrm{ind}_H(A) \leq 0$, then for $\lambda_0 \in H$ we choose an analytic function $f \in O(U, Z)$ on a neighbourhood of λ_0 as explained above. The analyticity of h on U follows from the observation that for $\lambda \in U$ and $y \in Y$

$$\langle y, C' f(\lambda)\rangle = h(\lambda)\langle y, f(\lambda)\rangle.$$

The argument in the case $\mathrm{ind}_H(A) > 0$ is analogous.

We used the eigenspaces of B, respectively A, to define the representing function h for a given operator C. The next result describes what happens if the eigenspaces are replaced by generalized eigenspaces. To formulate it we denote by Σ_H the set of discontinuity points of $\dim \mathrm{Ker}(\lambda - A)$ as a function of λ on H. It is well known that Σ_H is a discrete subset of H [11, Satz 104.4].

Proposition 1.2. *If* $\mathrm{ind}_H(A) \leq 0$, *then for* $\lambda \in H\backslash\Sigma_H$ *we have*

$$C'_{|\mathrm{Ker}(\lambda - B)^n} = \sum_{j=0}^{n-1} (h^{(j)}(\lambda)/j!)(B - \lambda)^j_{|\mathrm{Ker}(\lambda - B)^n}$$

for all integers $n \geq 1$. *In the case* $\mathrm{ind}_H(A) > 0$ *this formula remains valid if* C' *and* B *are replaced by* C *and* A.

Proof. We only consider the case $\operatorname{ind}_H(A) \le 0$. Let $\lambda_0 \in H\backslash\Sigma_H$ be fixed. By induction one can show that for each integer $n \ge 1$ and each $z \in \operatorname{Ker}(\lambda_0 - B)^n$ there are analytic functions $f_1, \dots, f_n \in O(\{\lambda_0\}, Z)$ with $(\lambda - B)f_i(\lambda) = 0$ near λ_0 and $z \in LH\{f_1(\lambda_0), f_2^{(1)}(\lambda_0), \dots, f_n^{(n-1)}(\lambda_0)\}$.

Although the case $n = 1$ is well known to the specialists, we indicate a possible proof. Since $\lambda_0 \in \rho_e(B)$, there is a finite-dimensional subspace M of Z such that $Z = (\lambda_0 - B)Z \oplus M$. Let $\{z_1, \dots, z_r\}$ be a basis for $\operatorname{Ker}(\lambda_0 - B)$. The map $(\lambda_0 - B, i): Z \oplus M \to Z$, where i denotes the inclusion map of M into Z, is onto for $\lambda = \lambda_0$. By [10, Lemma 1.7] one can choose analytic functions $g_1, \dots, g_r \in O(\{\lambda_0\}, Z \oplus M)$ with $(\lambda - B, i)g_i(\lambda) = 0$ near λ_0 and $g_i(\lambda_0) = z_i$. By [17, Lemma 2.1] the sequence

$$0 \to \mathbb{C}^r \overset{\psi(\lambda)}{\to} Z \oplus M \overset{(\lambda - B, i)}{\to} Z \to 0,$$

$$\psi(\lambda)(\alpha_i)_{i=1}^r = \sum_{i=1}^r \alpha_i g_i(\lambda),$$

is exact for λ near λ_0. If $\lambda_0 \notin \Sigma_H$, then $\operatorname{Im}(\lambda - B) \cap M = \{0\}$ for λ near λ_0, hence the functions g_i $(i = 1, \dots, r)$ have values in Z near λ_0.

Next we assume that the assertion is true for $n - 1$ and consider an element $z \in \operatorname{Ker}(\lambda_0 - B)^n \backslash \operatorname{Ker}(\lambda_0 - B)^{n-1}$. We choose an analytic function $f \in O(\{\lambda_0\}, Z)$ with $(\lambda - B)f(\lambda) = 0$ near λ_0 and $f(\lambda_0) = (B - \lambda_0)^{n-1}z$. Then

$$(B - \lambda)f^{(j)}(\lambda) = jf^{(j-1)}(\lambda)$$

for λ near λ_0 and $j \ge 1$ and, in particular,

$$(B - \lambda_0)^{n-1}\left(z - \frac{1}{(n-1)!}f^{(n-1)}(\lambda_0)\right) = 0.$$

The induction is completed by applying the induction hypothesis.

Let $f \in O(\{\lambda_0\}, Z)$ be a function with $(\lambda - B)f(\lambda) = 0$ and let $k \in \{0, \dots, n-1\}$. The observation that

$$\sum_{j=0}^{n-1}(h^j(\lambda)/j!)(B - \lambda)^j f^{(k)}(\lambda) = \sum_{j=0}^{k}\binom{k}{j}h^{(j)}(\lambda)f^{(k-j)}(\lambda) = C' f^{(k)}(\lambda)$$

holds for all λ, concludes the proof.

Let $T \in L(X)$ be a continuous linear operator on a complex Banach space X. As usual we denote by $\sigma_\delta(T)$ the defect spectrum of T, i.e., the set of all points $\lambda \in \mathbb{C}$ for which $\lambda - T$ is not onto. For a closed set F in $\mathbb{C}$ the spectral subspace of T belonging to F is by definition the linear space

$$X_T(F) = \{x \in X;\ x \in (z - T)O(\mathbb{C}\backslash F, X)\}.$$

Proposition 1.3. *For $M \subset \rho_e(A)$ arbitrary the set*

$$Y(M) = \bigcap((\lambda - A)^n Y;\ \lambda \in M,\ n \in \mathbb{N})$$

is a $\sigma(Y, Z)$-closed space in $\operatorname{Lat}(A)$ *with $\sigma_\delta(A|Y(M)) \subset \mathbb{C}\backslash M$.*

Proof. Since products of Fredholm operators are Fredholm, the spaces $(\lambda - A)^n Y$ occurring above are norm-closed, hence also $\sigma(Y, Z)$-closed [12, §33.4.(1)], invariant subspaces for A.

Since for any choice of pairwise distinct complex numbers $\lambda_1, \dots, \lambda_k$ and nonnegative integers $n_1, \dots, n_k$ the identity

$$(\lambda_1 - A)^{n_1} Y \cap \cdots \cap (\lambda_k - A)^{n_k} Y = (\lambda_1 - A)^{n_1} \cdot \cdots \cdot (\lambda_k - A)^{n_k} Y$$

holds [11, Aufgabe 80.5], we obtain the description

$$Y(M) = \bigcap((\lambda_1 - A) \cdot \cdots \cdot (\lambda_n - A)Y;\ n \in \mathbb{N},\ \lambda_1, \dots, \lambda_n \in M).$$

We fix a point $\lambda_0 \in M$ and define $N = \operatorname{Ker}(\lambda_0 - A)$. We denote by $\mathscr{F}$ the set of all functions $f\colon M \to \mathbb{N}$, which are equal to zero almost everywhere and set $Y_f = \bigcap_{\lambda \in M} (\lambda - A)^{f(\lambda)} Y$ for $f \in \mathscr{F}$. Since N is finite-dimensional, there is an element $f_0 \in \mathscr{F}$ with $Y_{f_0} \cap N = Y(M) \cap N$. If $y \in Y(M)$, then for each $f \in \mathscr{F}$ there is an element $x_f \in Y_f$ with $(\lambda_0 - A)x_f = y$. Since for $f \in \mathscr{F}$ with $f \geq f_0$

$$x_{f_0} = x_f + (x_{f_0} - x_f) \in Y_f + (Y_{f_0} \cap N) \subset Y_f,$$

we conclude that $x_{f_0} \in Y(M)$.

Using Proposition 1.3 it is easy to show that the spectral subspaces $X_T(F)$ of an operator $T \in L(X)$ belonging to a closed set F with $\sigma_e(T) \subset F$ are closed.

Corollary 1.4. *For each closed set F in $\mathbb{C}$ with $\sigma_e(A) \subset F$ the identity $Y_A(F) = Y(\mathbb{C}\backslash F)$ holds.*

Proof. It is well known that for an arbitrary closed set F in $\mathbb{C}$ the left space is contained in the right. The reason is, that for each function $f \in O(\mathbb{C}\backslash F, Y)$ for which $(\lambda - A)f(\lambda)$ is constant on $\mathbb{C}\backslash F$ and each $\mu \in \mathbb{C}\backslash F$, the unique function $g \in O(\mathbb{C}\backslash F, Y)$ with $g(\lambda) = (f(\lambda) - f(\mu))/(\mu - \lambda)$, $\lambda \neq \mu$, satisfies $(\lambda - A)g(\lambda) = f(\mu)$ on $\mathbb{C}\backslash F$.

In Proposition 1.3 we have shown that $\lambda - A\colon Y(\mathbb{C}\backslash F) \to Y(\mathbb{C}\backslash F)$ is onto for each $\lambda \in \mathbb{C}\backslash F$. Therefore (see, e.g., [13, Theorem 5.1] for each $y \in Y(\mathbb{C}\backslash F)$ there is an analytic function $f \in \sigma(\mathbb{C}\backslash F, Y)$ with $y = (\lambda - A)f(\lambda)$ on $\mathbb{C}\backslash F$.

If $T \in L(X)$ is a continuous linear operator on a complex Banach space X, then for each open set U in $\mathbb{C}$ we define $X_T(U) = \bigcup_K X_T(K)$, where K ranges over all compact subsets of U. As a consequence of Corollary 1.4 one obtains a duality relation between the spectral subspaces of A and B.

Corollary 1.5. *For each closed set F with $\sigma_e(A) \subset F$ we have*

$$Y_A(F) = {}^{\perp}Z_B(\mathbb{C}\backslash F).$$

Proof. Again it is well known that for an arbitrary closed set F in $\mathbb{C}$ the left space is contained in the right. The idea is the following. For $y \in Y_A(F)$ and $z \in Z_B(\mathbb{C}\backslash F)$ one can choose a compact subset K of $\mathbb{C}\backslash F$ and functions $f \in O(\mathbb{C}\backslash F, Y)$, $g \in O(\mathbb{C}\backslash K, Z)$ with $y = (\lambda - A)f(\lambda)$ on $\mathbb{C}\backslash F$, $z = (\lambda - B)g(\lambda)$ on $\mathbb{C}\backslash K$. If Γ is a cycle that surrounds K in $\mathbb{C}\backslash F$, then Cauchy's integral theorem implies that

$$\langle y, z\rangle = \frac{1}{2\pi i}\int_\Gamma \langle y, g(\lambda)\rangle\, d\lambda = \frac{1}{2\pi i}\int_\Gamma \langle f(\lambda), z\rangle\, d\lambda = 0.$$

If $F \supset \sigma_e(A)$, then using Corollary 1.4 one obtains

$$^{\perp}Z_B(\mathbb{C}\backslash F) \subset {}^{\perp}\left(\bigvee_{\substack{\lambda\in\mathbb{C}\backslash F \\ n\geq 1}} \operatorname{Ker}(\lambda - B)^n\right) = \bigcap_{\substack{\lambda\in\mathbb{C}\backslash F \\ n\geq 1}} \operatorname{Im}(\lambda - A)^n = Y_A(F).$$

Here the first inclusion follows from the fact that the generalized eigenspaces $\operatorname{Ker}(\lambda - B)^n$, $\lambda \in \mathbb{C}\backslash F$, $n \geq 1$, are contained in $Z_B(\{\lambda\})$.

The last two results together with Proposition 1.2 allow a precise description of the kernel of the representation Φ_H. As before let $H \subset \sigma(A)$ be a hole of $\sigma_e(A)$. We define $H_0 = H\backslash\Sigma_H$ and denote by $(A)'$ the commutant of A in $L(Y)$.

Theorem 1.6. (a) *If* $\operatorname{ind}_H(A) \leq 0$, *then* Φ_H *is a continuous algebra homomorphism with*

$$\operatorname{Ker}\Phi_H = \{C \in \sigma\text{-AlgLat}(A);\ CY \subset Y_A(\mathbb{C}\backslash H_0)\}.$$

For $C \in \sigma\text{-AlgLat}(A)$ *and each* $\sigma(Y, Z)$*-continuous operator* $U \in (A)'$ *we have* $\operatorname{Im}(CU - UC) \subset Y_A(\mathbb{C}\backslash H_0)$.

(b) *If* $\operatorname{ind}_H(A) > 0$, *then* Φ_H *is a continuous algebra homomorphism with*

$$\operatorname{Ker}\Phi_H = \{C \in \sigma\text{-AlgLat}(A);\ Y_A(H_0) \subset \operatorname{Ker} C\}.$$

For $C \in \sigma\text{-AlgLat}(A)$ *and each* $\sigma(Y, Z)$*-continuous operator* $U \in (A)'$ *we have* $Y_A(H_0) \subset \operatorname{Ker}(CU - UC)$.

Proof. (a) Assume that $\operatorname{ind}_H(A) \leq 0$. We fix an element $C \in \sigma\text{-AlgLat}(A)$ and set $h = \Phi_H(C)$. By Proposition 1.2 we know that $h = 0$ if and only if $\operatorname{Ker}(\lambda - B)^n \subset \operatorname{Ker} C'$ for all $\lambda \in H_0$ and $n \geq 1$ or, equivalently, $\operatorname{Im} C \subset \operatorname{Im}(\lambda - A)^n$ for all $\lambda \in H_0$ and $n \geq 1$. Therefore the claimed representation of $\operatorname{Ker}\Phi_H$ follows from Corollary 1.4. If $C \in \sigma\text{-AlgLat}(A)$ and $U \in (A)'$ is $\sigma(Y, Z)$-continuous, then again by Proposition 1.2 it follows that $\operatorname{Ker}(\lambda - B)^n \subset \operatorname{Ker}(CU - UC)'$ for all $\lambda \in H_0$ and $n \geq 1$.

(b) If $\operatorname{ind}_H(A) > 0$, then it suffices to apply part (a) to B and to use Corollary 1.5.

2. Main results

In the sequel we shall make the simplifying assumption that the index of A has the same sign on all holes of $\sigma_e(A)$. More precisely, we shall assume that $\operatorname{ind}_H(A) \leq 0$ for all holes H in $\sigma_e(A)$. We shall denote by U the set

$$U = \bigcup (H\backslash\Sigma_H;\ H \subset \sigma(A) \text{ is a hole of } \sigma_e(A)).$$

Our assumption is satisfied, for instance, if the operator A satisfies the single valued extension property, i.e., the map

$$O(W, Y) \to O(W, Y), \qquad f \to (z - A)f$$

is one-to-one for all open subsets W of $\mathbb{C}$; namely, in this case a result of Finch [9, Theorem 9] implies that $\operatorname{ind}_H(A) \leq -1$ for each hole H in $\sigma_e(A)$ contained in $\sigma(A)$. Moreover, since by general Fredholm theory either $H \subset \sigma_p(A)$ or

$\sigma_p(A) \cap H$ is discrete in H, the same result of Finch shows that in this case $\Sigma_H = \sigma_p(A) \cap H$.

Let S be a closed subset of $\mathbb{C}$. Recall that the operator A is said to possess Bishop's property (β) (modulo S), if the map

$$O(W, Y) \to O(W, Y), \qquad f \to (z - A)f$$

is injective with closed range for each open subset W of $\mathbb{C}$ (resp. of $\mathbb{C} \backslash S$). Finally, as usual, if K is a compact set in $\mathbb{C}$ and V is a bounded open set in $\mathbb{C}$, then we shall say that K is dominating in V if

$$\|f\|_{\infty, V} = \sup_{z \in K \cap V} |f(z)|$$

holds for all bounded analytic functions f on V. For operators satisfying Bishop's property (β) it was shown in [8] how to construct a representation of $\mathrm{AlgLat}(T)$ on the largest open set V in $\mathbb{C}$ in which the essential spectrum of T is dominating. Our next aim is to show that the "Fredholm representations" constructed in Lemma 1.1 and the "Scott Brown representations" constructed in [8] are compatible with each other.

Let S be a closed set in $\mathbb{C}$ such that our given operator $A \in L(Y)$ satisfies Bishop's property (β) modulo S, and let $V \subset \mathbb{C}$ be open such that $\sigma_e(A)$ is dominating in V and $S \cap V = \varnothing$ or $S \subset V$. In [1] (for the construction and notation, see [8, §1]) it was shown how to construct a canonical S-decomposable lifting $(Z, B) \overset{q}{\leftarrow} (X, T)$ for B. We recall that an operator $T \in L(X)$ on a Banach space X is called S-decomposable if for each open cover $\mathbb{C} = U_0 \cup \cdots \cup U_n$ with $S \subset U_0$ there are spaces $X_0, \dots, X_n \in \mathrm{Lat}(T)$ with

$$X = X_0 + \cdots + X_n, \quad \sigma(T|X_i) \subset U_i \qquad (i = 0, \dots, n).$$

The canonical lifting was used in §3 of [8] to construct a continuous algebra homomorphism

$$\Phi_V : \sigma\text{-AlgLat}(A) \to H^\infty(V).$$

Lemma 2.1. *If* $C \in \sigma$-$\mathrm{AlgLat}(A)$ *and* $H \subset \sigma(A)$ *is a hole in* $\sigma_e(A)$, *then*

$$\Phi_H(C)_{|H \cap V} = \Phi_V(C)_{|H \cap V}.$$

Proof. We fix a point $\lambda \in H \cap V$ and define $h = \Phi_H(C)$, $g = \Phi_V(C)$. By our assumption that $\mathrm{ind}_H(A) \le 0$ we can choose a nonzero vector $z \in \mathrm{Ker}(\lambda - B)$. The construction of the canonical lifting T (see [8, §1]) guarantees that there is a vector $x \in \mathrm{Ker}(\lambda - T)$ with $qx = z$. By Lemma 3.3 of [8] it follows that for all $y \in Y$

$$\langle y, h(\lambda)z \rangle = \langle Cy, qx \rangle = y \otimes x(g) = \langle y, qg(T|X_T(\{\lambda\}))x \rangle = \langle y, g(\lambda)z \rangle.$$

In view of the last lemma it is obvious that the representations Φ_H and Φ_V, where H runs through all holes in $\sigma_e(A)$ with $H \subset \sigma(A)$, can be glued together to give a continuous algebra homomorphism

$$\Phi: \sigma\text{-AlgLat}(A) \to H^\infty(\Omega),$$

where $\Omega = U \cup V$.

Theorem 2.2. *The map* Φ: σ-AlgLat$(A) \to H^\infty(\Omega)$ *is a continuous algebra homomorphism with*

$$\operatorname{Ker}\Phi = \{C \in \sigma\text{-AlgLat}(A)\,;\ \operatorname{Im} C \subset Y_A(\mathbb{C}\backslash\Omega)\}.$$

For $C \in \sigma$-AlgLat(A) *and each* $\sigma(Y, Z)$*-continuous operator* $U \in (A)'$ *we have*

$$\operatorname{Im}(CU - UC) \subset Y_A(\mathbb{C}\backslash\Omega).$$

Proof. Whenever $H \subset \sigma(A)$ is a hole of $\sigma_e(A)$ with $H_0 \cap (\mathbb{C}\backslash V) \neq \varnothing$, each nonempty component of $H_0 \cap V$ has a nontrivial intersection with $\mathbb{C}\backslash S$. This observation easily gives rise to the identity

$$Y_A(\mathbb{C}\backslash\Omega) = Y_A(\mathbb{C}\backslash U) \cap Y_A(\mathbb{C}\backslash V).$$

We fix an element $C \in \sigma$-AlgLat(A) as well as a $\sigma(Y, Z)$-continuous operator $U \in (A)'$ and define $g = \Phi(C)$. By Proposition 1.2 we know that $g_{|U} = 0$ if and only if $\operatorname{Ker}(\lambda - B)^n \subset \operatorname{Ker} C'$ for each $\lambda \in U$ and each $n \geq 1$ or, equivalently, if $\operatorname{Im} C \subset \operatorname{Im}(\lambda - A)^n$ for all $\lambda \in U$ and $n \geq 1$. Using Corollary 1.4 and Theorem 3.4 of [8] we obtain the claimed characterization of $\operatorname{Ker}\Phi$.

Similarly, by Proposition 1.2 it follows that

$$\operatorname{Ker}(\lambda - B)^n \subset \operatorname{Ker}(C'U' - U'C') \qquad (\lambda \in U,\ n \geq 1)$$

and hence that

$$\operatorname{Im}(CU - UC) \subset \bigcap_{\substack{\lambda\in U\\ n\geq 1}} \operatorname{Im}(\lambda - A)^n = Y_A(\mathbb{C}\backslash U).$$

The inclusion $\operatorname{Im}(CU - UC) \subset Y_A(\mathbb{C}\backslash V)$ follows from the proof of Lemma 3.7 of [8] (see also [8, proof of Theorem 3.4]).

As an application we obtain the results announced in the introduction. If A satisfies Bishop's property (β) globally, then V can be chosen as the largest bounded open set in $\mathbb{C}$, in which $\sigma_e(A)$ is dominating. The resulting set Ω is rather large in this case. More precisely,

$$\sigma(A) \cap (\mathbb{C}\backslash\Omega) \subset (\sigma_e(A) \cap (\mathbb{C}\backslash\Omega)) \cup (\sigma_p(A) \cap \rho_e(A)).$$

The first set $K = \sigma_e(A) \cap (\mathbb{C}\backslash\Omega)$ is a subset of $\partial\sigma_e(A)$, which is dominating in no open subset of $\mathbb{C}$. In particular, $R(K) = C(K)$ (cf. [3, Theorem 3]). The second set $N = \sigma_p(A) \cap \rho_e(A)$ is countable with all limit points contained in $\partial\sigma_e(A)$.

We recall from [1] that property (β) admits a dual characterization. A Banach space operator $T \in L(X)$ is said to possess property (δ), if the map

$$O(W)'\tilde{\otimes}X \to O(W)'\tilde{\otimes}X, \qquad u \to (z - T)u,$$

is onto for each open set W in $\mathbb{C}$ or, equivalently, if

$$X = X_T(\overline{U}_1) + \cdots + X_T(\overline{U}_n)$$

holds for each open cover $\mathbb{C} = U_1 \cup \cdots \cup U_n$ (see [1] for the equivalence and other characterizations).

Corollary 2.3. *Let* $R \in L(E)$ *be a continuous operator on a complex Banach space* E.

(a) *If* R *satisfies property* (β) *and*

$$\sigma_p(R) \cap \rho_e(R) = \varnothing, \qquad E_R(\partial\sigma_e(R)) = \{0\},$$

then $\mathrm{AlgLat}(R) \subset (R)''$.

(b) *If* R *satisfies property* (δ) *and*

$$\sigma_\delta(R) \cap \rho_e(R) = \varnothing, \qquad \overline{E_R(\mathbb{C}\backslash\partial\sigma_e(R))} = E,$$

then $\mathrm{AlgLat}(R) \subset (R)''$.

Proof. As in [8, §1] we define

$$Y = E, \quad Z = E', \quad A = R, \quad B = R'$$

in the setting of part (a) and

$$Y = E', \quad Z = E, \quad A = R', \quad B = R$$

in the setting of part (b). Since property (β) and property (δ) are completely dual to each other [1, §3], in both cases A satisfies (β) and B satisfies (δ). Since for each closed set F in $\mathbb{C}$

$$Y_A(F) = {}^{\perp}Z_B(\mathbb{C}\backslash F) \quad [8, \text{Lemma } 1.3],$$

we have in both cases the relation $Y_A(\sigma(A)\cap(\mathbb{C}\backslash\Omega)) = \{0\}$. Thus, the assertions follow from Theorem 2.2.

Of course, Corollary 2.3 becomes wrong without the conditions

$$\sigma_p(R) \cap \rho_e(R) = \varnothing \quad [\text{resp. } \sigma_\delta(R) \cap \rho_e(R) = \varnothing].$$

To see this, it suffices to recall that on a finite-dimensional space each operator R with $\mathrm{AlgLat}(R) \subset (R)'$ is reflexive [2].

Specialized to the case of hyponormal operators on Hilbert spaces we obtain the following consequences. We denote by λ the planar Lebesgue measure.

Corollary 2.4. *Let* A *be a hyponormal operator on a Hilbert space* H.

(a) *If* $\lambda(\partial\sigma_e(A)) = 0$, *then* $\mathrm{AlgLat}(A)$ *is commutative.*

(b) *If* A *is pure and* $\lambda(\partial\sigma_e(A)) = 0$, *then* $\mathrm{AlgLat}(A) \subset (A)''$.

Proof. Recall that hyponormal operators satisfy Bishop's property (β) [14, Theorem III.5.5]. If Ω is defined as above, then

$$\sigma(A|H_A(\mathbb{C}\backslash\Omega)) \subset \sigma(A) \cap (\mathbb{C}\backslash\Omega) \subset \partial\sigma_e(A) \cup N.$$

Since hyponormal operators, the spectrum of which is of Lebesgue measure zero, are normal, the space $M = H_A(\mathbb{C}\backslash\Omega)$ is a reducing subspace for A such that $A_{|M}$ is normal if $\lambda(\partial\sigma_e(A)) = 0$. Therefore, part (b) follows directly from Theorem 2.2. If $C, D \in \mathrm{AlgLat}(A)$, then

$$(CD - DC)(M^{\perp}) \subset M \cap M^{\perp} = \{0\}.$$

Moreover, $(CD - DC)(M) = \{0\}$, since $A_{|M}$ is reflexive as a normal operator [6, Theorem II.8.5].

Since the above methods are of a comparatively general nature, it is perhaps not surprising that in cases where there is much more structure at hand they do not lead to the best possible results. By a result of Olin and Thomson [15] all subnormal operators are reflexive. As an application of our methods we only obtain:

Corollary 2.5. *If A is a subnormal operator on a Hilbert space, then* AlgLat(A) *is commutative. If, in addition, A is pure, then* $\mathrm{AlgLat}(A) \subset (A)''$.

Proof. Since there is a reducing space H_0 for A such that $A_{|H_0}$ is normal and $A_{|H_0^\perp}$ is pure and subnormal [6, Proposition III.2.1], it suffices to prove the second statement. But, if A is pure, then $\sigma_p(A) = \varnothing$ and hence $\sigma(A|H_A(\mathbb{C}\backslash\Omega)) \subset \sigma_e(A) \cap (\mathbb{C}\backslash\Omega)$, where the last set $K = \sigma_e(A) \cap (\mathbb{C}\backslash\Omega)$ satisfies $R(K) = C(K)$. But then $A_{|H_A(\mathbb{C}\backslash\Omega)}$ is normal [6, Theorem VI.1.1] and hence $H_A(\mathbb{C}\backslash\Omega) = \{0\}$.

References

1. E. Albrecht and J. Eschmeier, *Functional models and local spectral theory*, preprint, Saarbrücken and Münster, 1991.
2. L. Brickman and P. A. Fillmore, *The invariant subspace lattice of a linear transformation*, Canad. J. Math. **19** (1967), 810–822.
3. S. W. Brown, *Hyponormal operators with thick spectrum have invariant subspaces*, Ann. of Math. (2) **125** (1987), 93–103.
4. B. Chevreau, G. Exner, and C. Pearcy, *On the structure of contraction operators*. III, Michigan Math. J. **36** (1989), 29–62.
5. I. Colojoară and C. Foiaş, *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.
6. J. B. Conway, *Subnormal operators*, Pitman, Boston, 1981.
7. J. A. Deddens, *Every isometry is reflexive*, Proc. Amer. Math. Soc. **28** (1971), 509–512.
8. J. Eschmeier and B. Prunaru, *Invariant subspaces for operators with Bishop's property* (β) *and thick spectrum*, J. Funct. Anal. **94** (1990), 196–222.
9. J. K. Finch, *The single valued extension property on a Banach space*, Pacific J. Math. **58** (1975), 61–69.
10. A. M. Gleason, *Finitely generated ideals in Banach algebras*, J. Math. Mech. **13** (1964), 125–132.
11. H. Heuser, *Funktionalanalysis*, 2.Auflage, B. G. Teubner, Stuttgart, 1986.
12. G. Köthe, *Topological vector spaces*. II, Springer-Verlag, Berlin, Heidelberg, and New York, 1979.
13. J. Leiterer, *Banach coherent analytic Fréchet sheaves*, Math. Nachr. **85** (1978), 91–109.
14. M. Martin and M. Putinar, *Lectures on hyponormal operators*, Operator Theory, Adv. Appl., vol. 39, Birkhäuser Verlag, Basel, Boston, and Berlin, 1989.
15. R. Olin and J. Thomson, *Algebras of subnormal operators*, J. Funct. Anal. **37** (1980), 271–301.
16. D. Sarason, *Invariant subspaces and unstarred operator algebras*, Pacific J. Math **17** (1966), 511–517.
17. J. L. Taylor, *A joint spectrum for several commuting operators*, J. Funct. Anal. **6** (1970), 172–191.

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