PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 118, Number 1, May 1993

THE ISOPERIMETRIC INEQUALITY FOR NONSIMPLE CLOSED CURVES

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. The main purpose of this paper is the generalization to the hyperbolic and elliptic spaces of the isoperimetric inequalilty of Banchoff and Pohl (J. Differential Geom. **6** (1971), 175–192).

1

Banchoff and Pohl [1] showed that the classical isoperimetric inequality still holds for nonsimple closed plane curves of length L, if the area is replaced by the sum of the areas into which the curve divides the plane, each weighted with the square of the winding number, i.e.,

$$L^2 - 4\Pi \int_{\mathbb{R}^2} w^2 \, dP \geq 0.$$

Equality holds only for one or several coincident circles, each traversed in the same direction a number of times. The authors also generalized the result to arbitrary dimension and codimension.

In this paper we show that a similar result is valid for compact oriented manifolds immersed in either the hyperbolic or the elliptic space. In §2 we obtain a generalization of formulae (4.7) and (4.8) of [1] in any n-dimensional Riemann-space of constant curvature by using some known formulae of integral geometry [3]. In §3 we show that the results of Banchoff and Pohl [1, §3] are valid without change in the hyperbolic and elliptic n-dimensional spaces. In §4 we obtain a general expression for the main result of Banchoff and Pohl [1], which can be stated as:

Let $\mathbb{W}^n$ denote the hyperbolic, elliptic, or euclidean n-dimensional space with $H(r)$ being $\operatorname{sh} r$, $\operatorname{sen} r$, or r, respectively. Let M be a compact oriented manifold of dimension m and $f\colon M \to \mathbb{W}^n$ be an immersion of class C^2. For $(x, y) \in M \times M$ let $r(x, y)$ denote the chord length from $f(x)$ to $f(y)$ and let dV_1, dV_2 denote the volume elements on M at x and y, respectively. Let $H_{n-m-1,n}$ denote the Grassmann manifold of all (unoriented) $(n-m-1)$-planes in $\mathbb{W}^n$ (parallel planes are not identified), and let $|dH_{n-m-1,n}|$ denote its invariant measure. For $h \in H_{n-m-1,n}$ let $\pm\lambda$ denote the linking number of

Received by the editors November 12, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53A35.

h with f. Then:

$$\int_{M\times M} H(r)^{-m+1}\, dV_1\, dV_2 - (1+m)\Sigma_m K_{m,n} \int_{H_{n-m-1,n}} \lambda^2 |dH_{n-m-1,n}| \tag{1.1}$$
$$+ \frac{(-1)^{m(m+1)/2}}{m!} \int_{G\cap M\times M\neq\varnothing} [H(r)-r]\, dG^m \geq 0,$$

where Σ_m denotes the surface volume of the unit m-sphere and $K_{m,n}$ is a constant, which depends only on m and n. Equality holds only for one or several coincident spheres with coincident orientations or $(m=1)$ one or several circles each traversed in the same direction a number of times.

For closed nonsimple plane curves we obtain

$$L^2 - 4\Pi \int_{\mathbb{W}^2} w^2\, dP - \int_{G\cap(M\times M)} [H(r)-r]\, d\vec{G} \geq 0,$$

where $d\vec{G}$ denotes the density for oriented geodesics.

This formula, when applied to simple closed curves, taking into account (18.10) and (18.11) of [3], gives the classical isoperimetric inequality:

$$L^2 - 4\Pi F + \varepsilon F^2 \geq 0,$$

where ε equals -1, 1, or 0 for the hyperbolic, elliptic, or euclidean plane, respectively.

2

Let $\mathbb{W}^n$ be a Riemann manifold of dimension n endowed with the fundamental quadratic form (for a local coordinate system) (see [3, p. 331])

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}\, dx_i\, dx_j.$$

Since the 2-form

$$dG = \sum_{i=1}^{n} dp_i \wedge dx_i$$

is an invariant form (any geodesic is defined by a point $(x_1, \dots, x_n)$ and a direction $(p_1, \dots, p_n)$), the exterior powers

$$dG^m = (dG)^m = m! \sum_{i_1<\cdots<i_m} dp_{i_1} \wedge dx_{i_1} \wedge \cdots \wedge dp_{i_m} \wedge dx_{i_m}$$

are also invariant forms $(m = 1, \dots, n-1)$ and they define a measure for $2m$-dimensional sets of geodesics. Consider the expression:

$$dG^m = m!(-1)^{m(m-1)/2} \sum_{i_1<\cdots<i_m} dp_{i_1} \wedge \cdots \wedge dp_{i_m} \wedge dx_i \wedge \cdots \wedge dx_{i_m} \tag{2.1}$$

for a family of geodesics parametrized in a special way; in particular, for geodesics that join the points of one m-dimensional submanifold of $\mathbb{W}^n$ to another. We restrict ourselves to $\mathbb{W}^n$ the n-dimensional hyperbolic, elliptic, or euclidean space, noting that they include any complete, simply-connected Riemann-space with constant curvature.

Let us consider as Pohl [2, p.1325] the following slightly more general situation:

Let M_1, M_2 be oriented m-dimensional differentiable manifolds of class C^1. Let $f: M_1 \to \mathbb{W}^n$ be an immersion, and let $F: M_1 \times M_2 \to \mathbb{W}^n$ be a continuously differentiable map such that for each fixed $x \in M_1$ the map $F_x(y) = F(x, y)$ describes an immersion of M_2. We assume as Pohl that $F(x, y) \neq f(x) \forall x, y$. Let G denote the geodesic that joins $f(x)$ to $F(x, y)$ oriented from $f(x)$ to $F(x, y)$. We are now going to prove equation (2.5) of [2] under these premises. For this purpose, let us define orthonormal frames $Xe_1, \dots, e_n$ and $Xa_1, \dots, a_n$ following Pohl [2, p.1325] with only the following modifications:

e_1 is the tangent unit vector to G in $f(x)$;
S_1 contains T_1 and e_1 (but not G); and
U_1 is the $(m+1)$-dimensional manifold containing M_1 and G, so that S_1 is the tangent space to U_1 in $f(x)$.

Since the orientations e_1e_2 and a_1a_2 agree on Q_1, we may write

$$de_2 = \operatorname{sen} \sigma_1 \, da_2. \tag{2.2}$$

We now determine G in U_1 by its intersection with S' $(e_1 = 0) \subset U_1$, i.e., X. Following Santaló [3, p. 334] we have:

$$ds^2 = \sum_{i=1}^{n} g_{ii}\, dx_i^2, \qquad p_i = (g_{ii})^{1/2} \cos \alpha_i, \qquad \alpha_i = \angle(G, e_i). \tag{2.3}$$

Using expression (2.1)

$$dG^m = m!(-1)^{(m-1)m/2} dp_2 \wedge \cdots \wedge dp_{m+1} \wedge de_2 \wedge \cdots \wedge de_{m+1}. \tag{2.4}$$

Also

$$du_m = \operatorname{sen} \alpha_2 \cdots \operatorname{sen} \alpha_{m+1} \, d\alpha_2 \wedge \cdots \wedge d\alpha_{m+1}, \tag{2.5}$$

where du_m denotes the area element on the unit m-dimensional sphere of center X corresponding to the direction of e_1 [3, p. 337] (notice that $\alpha = \angle(e_1, e_1)$). Since the volume element dV_1 on M at X has the value:

$$dV_1 = (g_{22} \cdots g_{m+1\,m+1})^{1/2} da_2 \wedge \cdots \wedge da_{m+1}, \tag{2.6}$$

replacing (2.2), (2.3), (2.5), and (2.6) in (2.4) we obtain

$$dG^m = m!(-1)^{m(m-1)/2} \operatorname{sen} \sigma_1 \, du_m \wedge dV_1. \tag{2.7}$$

Next we compute du_m with fixed x. To do this we translate the frame $Xe_1, \dots, e_n$ parallel along G to $F_x(y)$ obtaining $X'e'_1, \dots, e'_n$. Let S_2 be the $(m+1)$-dimensional linear space through X' containing e'_1 and T_2, the tangent space to F_x at y. We orient S_2 by e'_1T_2 unless e'_1 lies in T_2, in which case we take either orientation on S_2. Let $\sigma_2 = \angle(T_2, e'_1)$, $X'b_1, \dots, b_n$ be an orthonormal frame such that $b_1 = e'_1$; $b_2, \dots, b_{m+1}$ are in S_2 and so chosen that $b_1 \cdots b_{m+1}$ agrees with the given orientation of S_2.

We use polar coordinates in S_2, with respect to the frame $Xb'_1, \dots, b'_n$ (where b'_i is the parallel translated vector of b_i from X' to X along G). Therefore, we have

$$ds^2 = dr^2 + H(r)\, d\lambda_i^2$$

and for fixed r,

$$db_i = H(r)\,d\lambda_i\,, \qquad d\sigma_p = H(r)^m\,du'_m\,,$$

where $d\sigma_p$ denotes the m-dimensional volume element for fixed r, and du'_m is the area element on the unit m-dimensional euclidean sphere of center X corresponding to the direction of $b'_1 = e_1$ (defined for the frame $b'_1, \dots, b'_n$).

On the other hand if dV_2 denotes the volume element on M_2 at X',

$$d\sigma_p = dV_2 \operatorname{sen}\sigma_2\,,$$

and if we define $\cos\tau$ as Pohl [2, p. 1326] so that τ is interpreted as the angle between the oriented $(m+1)$-planes S_1 and S_2 (or its parallel translation along G) we have

$$\cos\tau = du_m/du'_m\,,$$

so that

$$du_m = H(r)^{-m}\cos\tau\operatorname{sen}\sigma_2\,dV_2.$$

Replacing in (2.7) we obtain

$$dG^m = m!(-1)^{m(m-1)/2}\operatorname{sen}\sigma_1\operatorname{sen}\sigma_2\cos\tau H(r)^{-m}\,dV_2\wedge dV_1\,, \tag{2.8}$$

which is one of the results of Pohl [2, p. 1326].

In order to obtain the general result we have to compute

$$\int_{M\times M} r\,dG^m = \int_{M\times M}(r\,dp'_2)\wedge da_2\wedge dG^{m-1}\,, \tag{2.9}$$

where $Xa_1, \dots, a_n$ is the frame defined above so that $p'_2 = (g_{22})^{1/2}\cos\sigma_1$. To do this we define as in [1] a certain 'secant space' and apply Stoke's theorem:

$$\int_{M\times M} r\,dp'_2 = r\cdot p'_2|_{r=0} - \int_{M\times M} p'_2\,dr = -\int (g_{22})^{1/2}\cos\sigma_1\,dr.$$

Since $\cos\sigma_2 = dr/db_2$ (for $Xb_1, \dots, b_n$ the frame defined above) we get:

$$\int_{M\times M} r\,dG^m = -\int_{M\times M}(g_{22})^{1/2}\cos\sigma_1\cos\sigma_2\,db_2\wedge da_2\wedge dG^{m-1}. \tag{2.10}$$

Defining frames as above and interpreting the angle ν as in [1] we have

$$du_{m-1} = \cos\nu\,du'_{m-1}$$

and therefore,

$$dG^{m-1} = (m-1)!(-1)^{m(m-1)/2}\cos\nu(g_{33}\cdots g_{m+1m+1})^{1/2}\,du'_{m-1} \wedge da_3\wedge\cdots\wedge da_{m+1}.$$

Since

$$dV_1 = (g_{22}\cdots g_{m+1\,m+1})^{1/2}\,da_2\wedge\cdots\wedge da_{m+1}$$

and

$$dV_2 = H(r)^{-m+1}\,db_2\wedge du'_{m-1}\,,$$

replacing in (2.10)

$$\int_{M\times M} r\,dG^m = -(m-1)!(-1)^{m(m-1)/2} \times\int_{M\times M} H(r)^{-m+1}\cos\sigma_1\cos\sigma_2\cos\nu\,dV_2\wedge dV_1\,, \tag{2.11}$$

which is the desired expression.

3

We now prove that the formulae of Banchoff and Pohl [1, §3] are valid without change in the elliptic and hyperbolic n-dimensional spaces. For this purpose we work in an analogous way to Banchoff and Pohl [1] but we make use of some known results of integral geometry [3].

We first consider the second integral in our result (see (1.1)) and write

$$\mathscr{A}(M) = K_{m,n} \int_{H_{n-m-1,n}} \lambda^2 |dH_{n-m-1,n}|, \tag{3.1}$$

where

$$K_{m,n} = \frac{\Sigma_{n-m-1} \cdots \Sigma_1}{\Sigma_n \cdots \Sigma_{m+2}} \qquad (n \geq m+2), \;\; K_{m,m+1} = 1,$$

is defined in Banchoff and Pohl [1, p. 180]. We now prove that $\mathscr{A}(M)$ is stable under rising of the codimension, i.e., if $f(M) \subseteq \mathbb{W}^n \subseteq \mathbb{W}^N$ then

$$K_{m,N} \int_{H_{N-m-1,N}} \lambda^2 |dH_{N-m-1,N}| = K_{m,n} \int_{H_{n-m-1,n}} \lambda^2 |dH_{n-m-1,n}|. \tag{3.2}$$

Comparing [3, (14.69)] firstly for $q \to n$, $r \to n-m$, $n \to n+1$, $M_q \to U_n$, and secondly for $q \to n$, $r \to n-m-1$, $n \to n$, $M_q \to U_n$, we obtain

$$\int_{H_{n-m,m+1} \cap U_n \neq \varnothing} dH_{n-m,m+1} = \frac{\Sigma_{n+1}}{\Sigma_{n-m}} \int_{H_{n-m-1,n} \cap U_n \neq \varnothing} dH_{n-m-1,n}. \tag{3.3}$$

On the other hand (see [2] or [3]),

$$dH_{n-m,n+1} = \bigwedge\nolimits_{j,\beta} \omega_{j\beta} \wedge \omega_\beta$$
$$(j = 1, \ldots, n-m; \;\; \beta = n-m+1, \ldots, n+1),$$
$$dH_{n-m-1,n} = \bigwedge\nolimits_{i,\alpha} \pi_{i\alpha} \wedge \pi_\alpha$$
$$(i = 1, \ldots, n-m-1; \;\; \alpha = n-m, \ldots, n)$$

with $\omega_{j\beta} = de_\beta \cdot e_j$, $\omega_\beta = dp \cdot e_\beta$, $\omega_{j\beta} + \omega_{\beta j} = 0$, for an orthonormal frame $pe_1, \ldots, e_{n+1}$ in p, and $\pi_{i\alpha} = da_\alpha \cdot a_i$, $\pi_\alpha = dp \cdot a_\alpha$, $\pi_{i\alpha} + \pi_{\alpha i} = 0$ for $pa_1, \ldots, a_{n+1}$ the frame defined as follows:

$$a_i = e_i, \quad i = 1, \ldots, n-m-1; \qquad a_{n-m} = e_{n+1};$$
$$a_{n-m+1} = e_{n-m+1}, \quad i = 1, \ldots, m; \qquad a_{n+1} = e_{n-m};$$

therefore,

$$|dH_{n-m,n+1}| = |dH_{n-m-1,n}| \wedge \bigwedge\nolimits_\alpha \pi_{n+1,\alpha}, \qquad \alpha = n-m, \ldots, n.$$

Since the linking number of an $(n-m)$-plane $h \subseteq \mathbb{W}^{n+1}$ with M does not change regarding h as lying in $\mathbb{W}^n$, using Fubini's theorem and integrating λ^2 we get

$$\int_{H_{n-m,n+1}} \lambda^2 |dH_{n-m,n+1}| = c_{n,m} \int_{H_{n-m-1,n}} \lambda^2 |dH_{n-m-1,n}|.$$

Applying (3.3) we obtain $c_{n,m} = \Sigma_{n+1}/\Sigma_{n-m}$. This proves the assertion for $N = n+1$, and so (3.2) is valid by induction.

Also since both the Cauchy formula and the remaining arguments used by Banchoff and Pohl [1] to demonstrate:

$$(3.4)\qquad \mathscr{A}(M^0) = K_{0,n}\int_{H_{n-1,n}} \lambda^2 |dH_{n-1,n}| = -1/2\Sigma_{i,j} r(x_i, x_j) i_i i_j$$

are also valid in the hyperbolic and elliptic spaces, equation (3.4) stays valid without change.

In order to prove that for $q > n-m-1$

$$(3.5)\qquad \mathscr{A}(M^m) = 1_{m,n,q}\int_{H_{q,n}} \mathscr{A}(M^m \cap H_q)|dH_{q,n}|$$

with

$$1_{m,n,q} = \frac{K_{m,n}\Sigma_{q-n+m}\cdots\Sigma_0\Sigma_{n-q-1}\cdots\Sigma_0}{K_{q-n+m,q}\Sigma_m\cdots\Sigma_0},$$

we use the fact that equations (12.36) and (12.52) of [3] are also valid in hyperbolic and elliptic spaces, so that

$$(3.6)\qquad \int_{\text{total}} dH_{r,n[q]} = \Sigma_{n-q-1}\cdots\Sigma_{n-r}/\Sigma_{r-q-1}\cdots\Sigma_0,$$

$$(3.7)\qquad dH_{i+1,r}\wedge dH^*_{r,n} = dH_{r,n[i+1]}\wedge dH_{i+1,n},$$

where $dH_{r,n[i+1]}$ is the density of $H_{r,n}$ about $H_{i+1,r}$ and $H^*_{r,n}$ denote the oriented Grassmannians. Using that, for

$$(3.8)\qquad H_{n-m-1,q}\subseteq H_{q,n},$$

it is

$$\lambda(M^m, h_{n-m-1}) = \lambda(M^m\cap h_q, h_{n-m-1}),$$

we now integrate λ^2 over the set of pairs of linear spaces $(H_{q,n}; H_{n-m-1,q})$ so that (3.8) is valid, and apply (3.7) to obtain

$$\int \lambda^2\, dH_{n-m-1,q}\wedge dH^*_{q,n} = \int \lambda^2\, dH_{q,n[n-m-1]}\wedge dH_{n-m-1,n}.$$

Evaluating the right-hand side using equation (3.6) for $r\to q$, $q\to n-m-1$; and taking into account the definition (3.1) in both sides, we compare and get equation (3.5) after few calculations.

Comparing as above, the densities $dH_{n-m,n}$ and $dH_{n-m-1,n-1}$, using orthonormal frames $pe_1,\ldots,e_n$ and $pa_1,\ldots,a_n$ with $a_1=e_1$; $a_i = e_{n-m-1+i}$ $i=2,\ldots,m+1$; $a_{m+j}=e_j$ $j=2,\ldots,n-m$; and using:

$$|dG^m|/m! = |\pi_{12}\wedge\cdots\wedge\pi_{1m+1}\wedge\pi_2\wedge\cdots\wedge\pi_{m+1}|$$

(which is easy to prove from expression (2.1)), we obtain

$$|dH_{n-m,n}| = |dH_{n-m-1,n-1[0]}|\wedge(1/m!)|dG^m|.$$

We now integrate the function $f(r)=r$ using Fubini's theorem to get

$$(3.9)\qquad \int r|dH_{n-m,n}| = c_{m,n}/m!\int r|dG^m|.$$

Using a general result of differential geometry (see [2, p. 1328]) and a particular case of (3.5) combined with (3.4), and evaluating the constant $c_{m,n}$ as Banchoff and Pohl [1, p. 186], equation (3.9) yields

$$\mathscr{A}(M^m) = \frac{(-1)^{1+m(m+1)/2}}{m!\Sigma_m} \int_{M\times M} r\,(dG^m)\,, \tag{3.10}$$

which is Theorem 4 of [1].

4

In order to obtain the general result we are now able to use the same arguments as Banchoff and Pohl [1] with $H(r)$ instead of r. Using the results of §§2 and 3, we obtain

$$\int H(r)^{-m+1}\,dV_1\,dV_2 - (1+m)\Sigma_m\mathscr{A}(M) + \frac{(-1)^{m(m+1)/2}}{m!}\int [H(r)-r]\,dG^m \geq 0$$

with equality holding for one or several coincident spheres with coincident orientations or for $(m=1)$ one or several coincident circles all transversed in the same direction each a number of times. Applied to the euclidean space $(H(r)=r)$, it yields the main theorem of Banchoff and Pohl [1].

References

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