AN UNCOUNTABLE COLLECTION
OF MUTUALLY INCOMPARABLE CHAINABLE CONTINUA

MARWAN M. AWARTANI

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ABSTRACT. We exhibit the existence of continuum many compactifications of
the ray with the arc as remainder, no one of which maps onto any other. We
also obtain continuum many pathwise connected circularly chainable continua
no one of which maps onto any other. The involved constructions and proofs
are combinatorial in nature.

1. INTRODUCTION

In [8] Rogers raised the question: "Does there exist an uncountable collection
of chainable continua, no member of which maps onto any other?" In [3]
Bellamy gave an affirmative answer to this question. However each member of
Bellamy's collection consisted of infinitely many path components. The purpose
of this paper is to prove the existence of the desired collection within the family
of compactifications of the the ray with the arc as remainder, whose members
consist of exactly two path components. This also generalizes the main result
in [1]. For related work see [2, 6, 7].

As a corollary, we obtain the known result [3, 5, 9] that there exists an un-
countable collection of circularly chainable continua, no member of which maps
onto any other. Unlike the continua in [3, 5, 9], which consist of infinitely many
path components, each member of this collection is pathwise connected.

First we define a partial order on the class of all sequences of zeroes and
ones and prove that with respect to the given partial order, this class contains
uncountably many mutually incomparable elements. Then we associate with
each sequence of zeroes and ones a compactification of the ray with the arc as
remainder. Finally we prove that if a surjective map exists between a pair of
such compactifications then the underlying sequences are comparable.

2. Preliminary results

N denotes the set of natural numbers.

2.1. Definition. Let 2^N denote the set of all sequences of zeroes and ones, and let $\{a_i\}$, $\{b_i\}$ be two elements in 2^N. Then

  (i) $\{a_i\}$ is said to vertically dominate $\{b_i\}$ if $a_i \geq b_i$ eventually.
  (ii) $\{a_i\}$ is said to dominate $\{b_i\}$ if there exists an integer j_0 such that $a_i \geq b_{i+j_0}$ eventually.
  (iii) $\{a_i\}$ and $\{b_i\}$ are called incomparable if neither of them dominates the other.

The following lemma is easy to verify.

2.2. Lemma. *Let $\{a_i\}$, $\{b_i\}$ be two elements in 2^N. If $\{a_{i_j}\}_{j=1}^{\infty}$ is a tail of the sequence of zeroes in $\{a_i\}$, and if for some integer j_0, $b_{i_j + j_0}$ is eventually zero, then $\{a_i\}$ dominates $\{b_i\}$.*

2.3. Lemma. 2^N *contains continuum many elements no one of which vertically dominates any other.*

Proof. Let $P(N)$ denote the power set of N. For each $\alpha = \{\alpha_i\} \in 2^N$ define $F(\alpha) \in P(N)$ as follows: $i \in F(\alpha)$ iff $\alpha_i = 1$. F is clearly a bijection. Moreover, if $F(\alpha)$ and $F(\beta)$ are infinite and almost disjoint (i.e., $F(\alpha) \cap F(\beta)$ is finite) then α and β are vertically incomparable. The result now follows since it is well known that $P(N)$ contains a collection of cardinality c consisting of mutually almost disjoint infinite subsets of N.

2.4. Lemma. 2^N *contains continuum many elements none of which is eventually constant and no one of which dominates any other.*

Proof. For each $\alpha = \{\alpha_i\} \in 2^N$ define a sequence $\{a_n^\alpha\}$ as follows:

$$a_n^\alpha = 0, \quad n \neq 2^j \text{ or } 2^j + 1;$$
$$a_n^\alpha = 1, \quad n = 2^j;$$
$$a_n^\alpha = \alpha_j, \quad n = 2^j + 1.$$

It is obvious that if $\{a_n^\alpha\}$ dominates $\{a_n^\beta\}$ then α vertically dominates β. The result now follows from Lemma 2.3.

3. Constructing the uncountable collection

3.1. *Notation.* Throughout this paper, if $f\colon (0, 1] \to [0, 1]$ is a continuous function, then the points of the graph of f are ordered by their first coordinate. Moreover if a and b are two points in the graph, then (a, b), $[a, b]$ denote respectively the open and closed portion of the graph between a and b. If A is a subarc of the graph, then $d(A)$ denotes the length of A measured along the graph. Π_1 and Π_2 denote the standard projections on the first and second coordinates respectively. All maps are assumed to be continuous.

3.2. Construction. With each $\alpha = \{\alpha_i\} \in 2^N$, we associate a compactification E_α of the ray J_α with the arc I_α as remainder. It suffices to describe J_α.
  (i) J_α is the graph of a piecewise linear function from $(0, 1]$ to $[0, 1]$.

(ii) The point $(1, 1/2) \in J_\alpha$.

(iii) Let m_α and M_α denote respectively the set of minima and the set of maxima of J_α and let $V_\alpha = m_\alpha \cup M_\alpha$.

(a) If $\alpha_i = 0$ then $[a_i, b_i] \cap V_\alpha$ has the following properties: (1) $|[a_i, b_i] \cap M_\alpha| = 2i$ and $|[a_i, b_i] \cap m_\alpha| = 2i - 1$. (2) If $\{M_{i,j} : 1 \leq j \leq 2i\}$ is an enumeration from right to left of the elements of the set $[a_i, b_i] \cap M_\alpha$ then $\pi_2(M_{i,2i}) = 1$ and $\pi_2(M_{i,j}) = \pi_2(M_{i,2i-j}) = (j+2)/(j+3) \, \forall j, \ 1 \leq j \leq i$. (3) $\pi_2([a_i, b_i] \cap m_\alpha) = 1/2$.

(b) If $\alpha_i = 1$ then $[a_i, b_i] \cap V_\alpha$ has the following properties: (1) $|[a_i, b_i] \cap M_\alpha| = 2i + 1$ and $|[a_i, b_i] \cap m_\alpha| = 2i$. (2) If $\{M_{i,j} : 1 \leq j \leq 2i + 1\}$ is an enumeration from right to left of the elements of the set $[a_i, b_i] \cap M_\alpha$ then $\pi_2(M_{i,2i+1}) = 1$ and $\pi_2(M_{i,j}) = \pi_2(M_{i,2i+1-j}) = (j+2)/(j+3) \forall j, 1 \leq j \leq i$. (3) If $\{m_{i,j} : 1 \leq j \leq 2i\}$ is an enumeration from right to left of the elements of the set $[a_i, b_i] \cap m_\alpha$ then $\pi_2(m_{i,j}) = 2/3$ for $j = i$ and $1/2$ for $j \neq i$.

(c) For each $i \in N$, $[b_i, a_i] \cap V_\alpha$ has the following properties: (1) $|[b_i, a_i] \cap M_\alpha| = 2i - 1$ and $|[b_i, a_i] \cap m_\alpha| = 2i$. (2) $\pi_2([b_i, a_i] \cap M_\alpha) = 1/2$. (3) If $\{m_{i,j} : 1 \leq j \leq 2i\}$ is an enumeration from right to left of the elements of the set $[b_i, a_i] \cap m_\alpha$ then $\pi_2(m_{i,2i}) = 0$ and $\pi_2(m_{i,j}) = \pi_2(m_{i,2i-j}) = 1 - (j+2)/(j+3) \forall j, 1 \leq j \leq i$.

3.3. *Notation.* Unless mentioned otherwise, the following notations are used for the above-constructed compactification E_α.

(i) Let $a_\alpha, b_\beta, c_\alpha$ denote respectively the points $(0, 0), (0, 1/2)$, and $(0, 1)$ in I_α.

(ii) For each $i \in N$ let K_i^α and L_i^α denote the arcs $[a_i, b_i]$ and $[b_i, a_{i+1}]$ respectively.

(iii) For each $i \in N$ let u_i^α denote the arc in J_α joining the last element of $L_i^\alpha \cap m_\alpha$ with the first element of $K_{i+1}^\alpha \cap M_\alpha$. Similarly, let w_i^α denote the arc in J_α joining the last element of $K_i^\alpha \cap M_\alpha$ with the first element of $L_i^\alpha \cap m_\alpha$. The following lemma is easy to verify.

3.4. **Lemma.** *For a compactification E_α the following hold:*

(i) *If $t \in (a_\alpha, c_\alpha)$ and $t' \in I_\alpha$ then there exist sequences $\{t_i\}$ and $\{t_i'\}$ in J_α converging to t and t' respectively such that $\{d[t_i, t_i']\}$ is bounded away from infinity.*

(ii) *Let $\{t_i\}$ and $\{t_i'\}$ be two sequences in J_α converging to a_α and c_α respectively; then $\lim d[t_i, t_i'] = \infty$.*

(iii) *For each $i \in N$ let $p_i \in L_i^\alpha$; then $\lim d[p_i, p_{i+1}] = \infty$.*

(iv) *Let $\{p_i\}$ and $\{q_i\}$ be two sequences in J_α both converging to a_α. If $\{d[p_i, q_i]\}$ is bounded away from zero then $[a_\alpha, b_\alpha] \subseteq \lim[p_i, q_i]$.*

(v) *Let $\{A_i\}$ be a sequence of arcs in J_α such that $A_i \cap w_{j_i}^\alpha \neq \varnothing$ eventually. If $\lim A_i \supset [a_\alpha, b_\alpha]$ then $\lim d(A_i) = \infty$.*

(vi) *Let $p \in u_{k_i}^\alpha$ and $q \in w_{k_i+1}^\alpha$, then $[p, q] \cap m_\alpha \cap (y = 2/3) = \varnothing$ iff $\alpha_{k_i} = 0$.*

4. THE MAIN RESULTS

4.1. **Lemma.** *Let $h: E_\alpha \to E_\beta$ be a surjective map, and let A be a closed arc in J_α. If B is a closed arc in $h(A)$ then there exists a closed arc in A whose image is B.*

Proof. The result follows directly since the map $h|A \colon A \to h(A)$ is simply a map between closed intervals.

4.2. Lemma. *Let* $h \colon E_\alpha \to E_\beta$ *be a surjective map, and let* $\{A_i\}$ *be a sequence of arcs in* J_α *such that* $\limsup A_i \subseteq I_\alpha$. *The following hold.*

    (i) *If* $\{d(h(A_i))\}$ *is bounded away from zero then* $\{d(A_i)\}$ *is bounded away from zero.*

    (ii) *If* $\{d(A_i)\}$ *is bounded away from infinity then* $\{d(h(A_i))\}$ *is bounded away from infinity.*

Proof. (i) We may assume without loss of generality that $\{A_i\}$ is a convergent sequence. Since $\{d(h(A_i))\}$ is bounded away from zero, each $h(A_i)$ contains a pair of points p_i, q_i such that $\lim p_i \neq \lim q_i$. Hence A_i contains a pair of points p_i', q_i' such that $\lim p_i' \neq \lim q_i'$, which implies that $\{d(A_i)\}$ is bounded away from zero.

(ii) Suppose that $\lim d(h(A_i)) = \infty$. Then for each $i \in N$, $h(A_i)$ contains a collection $\{B_{i,j} \colon 1 \leq j \leq i\}$ of disjoint closed subarcs such that $\{d(B_{i,j}) \colon (i,j) \in N \times \{1, 2, \ldots, i\}\}$ is bounded away from zero. For each $i \in N$ and each j, $1 \leq j \leq i$, let $A_{i,j}$ be a subarc of A_i such that $h(A_{i,j}) = B_{i,j}$. This is possible by Lemma 4.1. It follows from part (i) that $\{d(A_{i,j}) \colon (i,j) \in N \times \{1, \ldots, i\}\}$ is bounded away from zero, implying that $\lim d(A_i) = \infty$.

4.3. Lemma. *Let* $h \colon E_\alpha \to E_\beta$ *be a surjective map. Then the following hold.*

    (i) $h^{-1}\{a_\beta, c_\beta\} = \{a_\alpha, c\alpha\}$.

    (ii) *If* $h(a_\alpha) = a_\beta$, *then* $h[a_\alpha, b_\alpha] = [a_\beta, b_\beta]$ *and* $h[b_\alpha, c_\alpha] = [b_\beta, c_\beta]$.

    (iii) *If* $h(a_\alpha) = c_\beta$, *then* $h[a_\alpha, b_\beta] = [c_\beta, b_\beta]$ *and* $h[b_\alpha, c_\alpha] = [b_\beta, a_\beta]$.

    (iv) $h(b_\alpha) = b_\beta$.

Proof. (i) It suffices to prove that $(a_\alpha, c_\alpha) \cap h^{-1}(c_\beta) = \varnothing = (a_\alpha, c_\alpha) \cap h^{-1}(a_\beta)$. Suppose that $(a_\alpha, c_\alpha) \cap h^{-1}(a_\beta) \neq \varnothing$, and let $t' \in [a_\alpha, c_\alpha]$, $t \in (a_\alpha, c_\alpha)$, be chosen so that $h(t') = c_\beta$ and $h(t) = a_\beta$. By Lemma 3.4(i) we can find sequences $\{t_i\}$ and $\{t_i'\}$ in J_α such that $\lim t_i = t$ and $\lim t_i' = t'$ where $\{d[t_i, t_i']\}$ is bounded away from infinity. Since $\lim h(t_i) = a_\beta$ and $\lim h(t_i') = c_\beta$, it follows from Lemma 3.4(ii) that $\lim d[h(t_i), h(t_i')] = \infty$. This contradicts Lemma 4.2(ii). It can be similarly shown that $h^{-1}(c_\beta) \cap (a_\alpha, c_\alpha) = \varnothing$.

(ii) We only prove that $h[b_\alpha, c_\alpha] = [b_\beta, c_\beta]$ since the proof that $h[a_\alpha, b_\alpha] = [a_\beta, b_\beta]$ is similar and thus omitted. Since $h(a_\alpha) = a_\beta$, (i) implies that $h^{-1}(c_\beta) = c_\alpha$. Suppose that there exists $t \in (b_\alpha, c_\alpha)$ such that $h(t) < b_\beta$. Let T denote the sequence $J_\alpha \cap (y = t)$. Two cases arise:

Case (a). There exist two subsequences (p_i) and (q_i) of T such that for each $i \in N$: (1) p_i and q_i are adjacent in T, (2) $\{p_i, q_i\} \subseteq K_{j_i}^\alpha$, and (3) $h(p_i) \in L_{k_i}^\beta$ and $h(q_i) \in L_{r_i}^\beta$ with $r_i \neq k_i$. It follows from Lemma 3.4(iii) that $\lim d[h(p_i), h(q_i)] = \infty$ and consequently that $\lim(h[p_i, q_i]) = \infty$. This contradicts Lemma 4.2(ii) since $\{d[p_i, q_i]\}$ is bounded away from infinity.

Case (b). $h(K_i^\alpha \cap T) \subseteq L_{j_i}^\beta \ \forall i > i_0$. Then there exists a subsequence $\{K_r^\alpha\}_{i=1}^\infty$ such that $h(K_{r_i}^\alpha \cap T) \subseteq L_{j_i}^\beta$ where $j_i \neq j_k$ whenever $i \neq k$. For each $i \in N$ let $p_{r_i}^\cdot$, q_{r_i} be the first and last points respectively of $k_{r_i}^\alpha \cap T$. Since $L_{j_i}^\beta \cap L_{j_{i+1}} = \varnothing$,

it follows that $h[q_{r_i}, p_{r_{i+1}}] \supset K_{l_i}^\beta$ for some $l_i \in N$, which in turn implies that $h(\lim[q_{r_i}, p_{r_{i+1}}]) \supset \lim K_{l_i}^\beta$. That is, $c_\beta \in h[a_\alpha, t]$ with $t < c_\alpha$, contradicting our assumption that $h^{-1}(c_\beta) = c_\alpha$. The proof of (iii) is similar and thus omitted and the proof of (iv) follows directly from (ii) and (iii) above.

4.4. Lemma. *Let $h: E_\alpha \to E_\beta$ be a surjective map. If neither α nor β is eventually constant then the following hold.*

  (i) $h(a_\alpha) = a_\beta$; $h(c_\alpha) = c_\beta$; $h[a_\alpha, b_\alpha] = [a_\beta, b_\beta]$; $h[b_\alpha, c_\alpha] = [b_\beta, c_\beta]$.
  (ii) *If $\{A_i\}$ is a sequence of arcs in J_α such that $\lim A_i = [b_\alpha, c_\alpha]$ and $A_i \cap m_\alpha \cap (y = 2/3) = \varnothing$, $\forall i \in N$, then $h(A_i) \cap m_\beta \cap (y = 2/3) = \varnothing$ eventually.*

Proof. (i) By Lemma 4.3(i), it suffices to prove that $h(a_\alpha) = a_\beta$. Suppose that $h(a_\alpha) = c_\beta$. It follows from Construction 3.2 that there exist sequences $\{p_i\}$ and $\{q_i\}$ in J_β such that $\lim p_i = \lim q_i = c_\beta$, $\lim[p_i, q_i] = [2/3, c_\beta]$, and $\{d[p_i, q_i]\}$ is bounded away from zero. By Lemma 4.1 we can find a sequence $\{B_i\}$ of arcs in J_α such that $h(B_i) = [p_i, q_i]$. Since $\{d[p_i, q_i]\}$ is bounded away from zero, it follows from Lemma 4.2(i) that $\{d(B_i)\}$ is bounded away from zero. Choose p_i and q_i in B_i such that $h(p_i) = p_i$ and $h(q_i) = q_i$. Since $h^{-1}(c_\beta) = a_\alpha$, it follows that $\lim p_i = \lim q_i = a_\alpha$. We may assume without loss of generality that $\{B_i\}$ is a convergent sequence. Hence by Lemma 3.4(iv), $[a_\alpha, b_\alpha] \subseteq \lim B_i$. Since $\lim h(B_i) = [2/3, c_\beta]$, it follows that $h[a_\alpha, b_\alpha] \supset [2/3, c_\beta]$, contradicting Lemma 4.3(iv). Hence $h(c_\alpha) = c_\beta$ and $h(a_\alpha) = a_\beta$. The proof of (ii) is similar and thus omitted.

4.5. Lemma. *Let $h: E_\alpha \to E_\beta$ be a surjective map, and let $\{p_i\}$ and $\{q_i\}$ be sequences of points in J_α such that for each $i \in N$, $p_i \in u_i^\alpha$ and $q_i \in w_i^\alpha$. If $\lim q_i$ and neither α nor β is eventually constant, then there exists an integer j_0 such that eventually $h(p_i) \in u_{i+j_0}^\beta$ and $h(q_i) \in w_{i+j_0}^\beta$.*

Proof. Since $h(b_\alpha) = b_\beta$, it follows that $\lim h(p_i) = \lim h(q_i) = b_\beta$. We break the proof into four steps:

Step 1. $h(p_i) \in \bigcup_{j \in N} u_j^\beta$ eventually. Suppose not, then three cases arise:

Case (i). $\{d[h(p_i), M_{j_i}]\}$ converges to zero for some subsequences $\{M_{j_i}\}_{i=1}^\infty$ of M_β. Then it can be shown using the uniform continuity of h that $\exists \delta > 0$, $\ni h[b_\alpha, b_\alpha + \delta] \subseteq [a_\beta, b_\beta]$ contradicting Lemma 4.4(i).

Case (ii). $\{d[h(p_i), m_{j_i}]\}$ converges to zero for some subsequence $\{m_{j_i}\}$ of m_β. Using a reasoning similar to the above, it can be shown that $\exists \delta > 0$, $\ni h[b_\alpha, b_\alpha - \delta] \subseteq [b_\beta, c_\beta]$ again contradicting Lemma 4.4(i).

Case (iii). $h(p_i) \in \bigcup_{j \in N} w_j^\beta$ eventually. Let $m_i^* = u_i^\alpha \cap m_\alpha$. Since $\lim[m_i^*, p_i] = [a_\alpha, b_\alpha]$, Lemma 4.4 implies that

$$\lim h[m_i^*, p_i] = [a_\beta, b_\beta].$$

Since $h[m_i^*, p_i] \cap (\bigcup_{j \in N} w_j^\beta) \neq \varnothing$ eventually, it follows from Lemma 3.4(v) that $\lim d(h[m_i^*, p_i]) = \infty$. This contradicts Lemma 4.2(ii) since $\{d[m_i^*, p_i]\}$ is bounded away from infinity. Hence the only remaining possibility is for $h(p_i)$ to be eventually in $\bigcup_{j \in N} u_j^\beta$.

Step 2. $h(q_i) \in \bigcup_{j \in N} w_j^\beta$ eventually. The proof is similar to that in Step 1 and thus omitted.

Step 3. If $h(p_i) \in u_{j_i}^\beta$ eventually, then $h(q_i) \in w_{j_i}^\beta$ eventually. Suppose not, then $h(q_i) \in w_{k_i}^\beta$ where $k_i \neq j_i$ infinitely often. Then it can be deduced that $h[a_{,\alpha}, b_\alpha] \cap (b_\beta, c_\beta) \neq \varnothing$ contradicting Lemma 4.4(i).

Step 4. We finally prove that if $h(p_i) \in u_{j_i}^\beta$ eventually then $j_{i+1} = j_i + 1$. Suppose that $j_{i+1} \neq j_i + 1$ infinitely often; then it can be deduced that $h[b_\alpha, c_\alpha] \cap (a_\beta, b_\beta) \neq \varnothing$ contradicting Lemma 4.4(i).

It follows from the above that there exists an integer j_0 such that $h(p_i) \in u_{i+j_0}^\beta$ and $h(q_i) \in w_{i+1_0}^\beta$ eventually. This completes the proof of the lemma.

4.6. Theorem. *Let $h: E_\alpha \to E_\beta$ be a surjective map. If neither α nor β is eventually constant then α dominates β.*

Proof. For each $i \in N$ choose $p_i \in w_i^\alpha$ and $q_i \in w_i^\alpha$ such that $\lim p_i = \lim q_i = b_\alpha$. By Lemma 4.5, there exists an integer j_0 such that $h(p_i) \in u_{i+j_0}^\beta$ and $h(q_i) \in w_{i+j_0}^\beta$ eventually. Let $\{a_{k_i}\}$ be a tail of the sequence of all zeros in α. By Lemma 2.2, it suffices to prove that $\beta_{k_i+j_0} = 0$ eventually. Since $\alpha_{k_i} = 0$, it follows from Lemma 3.4(vi) that $[p_{k_i}, q_{k_i}] \cap m_\alpha \cap (y = 2/3) = \varnothing$. By Lemma 4.4(ii), $h[p_{k_i}, q_{k_i}] \cap m_\beta \cap (y = 2/3) = \varnothing$ eventually and hence $[h(p_{k_i}), h(q_{k_i})] \cap (y = 2/3) = \varnothing$ eventually. Since $h(p) \in u_{k_i+j_0}^\beta$ and $h(q_{k_i}) \in w_{k_i+j_0}^\beta$, we conclude by Lemma 3.4(vi) that $\beta_{k_i+j_0} = 0$ evenually, which completes the proof of the theorem.

Combining Lemma 2.4 and Theorem 4.6, we readily obtain the main result.

4.7. Theorem. *There exists an uncountable collection of cardinality c of compactifications of the ray with the arc as remainder, no member of which maps onto any other.*

4.8. Corollary. *There exists an uncountable collection of planar pathwise connected circularly chainable continua, no member of which maps onto any other.*

Proof. For each $\alpha \in 2^N$ let E_α be the associated compactification of the ray described in Construction 3.2. From E_α we obtain a circularly chainable continuum E_α' by simply trying the first point of J_α with the point $a_\alpha \in I_\alpha$. E_α' is clearly pathwise connected and planar. Notice that $E_\alpha' - (a_\alpha, c_\alpha]$ is a ray densely embedded in E_α. If $h: E_\alpha' \to E_\beta'$ is a surjective map then it is easy to verify that there exists a subray J_α' of $E_\alpha' - (a_\alpha, c_\alpha]$ having the following properties:

(i) If $\overline{J}_\alpha'$ denotes the closure of J_α' in E_α' then $\overline{J}_\alpha' - J_\alpha' = I_\alpha$.
(ii) $h(J_\alpha') = J_\beta$.

Consequently $h|\overline{J}_\alpha': \overline{J}_\alpha' \to \overline{J}_\beta$ is a surjective map for which the preceding results in 4.1–4.6 still hold from Theorem 2.4.

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DEPARTMENT OF MATHEMATICS, BIRZEIT UNIVERSITY, BIRZEIT, WEST BANK