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FREE SUBGROUPS OF QUATERNION ALGEBRAS

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ABSTRACT. Using the theory of group actions on trees, we shall prove that if a quaternion algebra over Laurant polynomials is not split then a certain congruence subgroup of the group of norm one elements is a free group. This generalizes and gives an easy, conceptually simpler proof than that given by Pollen for the field of real numbers.

Suppose that F is a field; with elements α and β we construct the quaternion algebra $\mathscr{H}$ over F with basis $1, I, J$, and K so that $I^2 = \alpha$, $J^2 = \beta$, and $K = IJ = -JI$. The involution σ on $\mathscr{H}$ is defined by $I \to -I$, $J \to -J$, $K \to -K$ and extended linearly to all of $\mathscr{H}$ by the identity on F. The norm $\mathscr{N}$ is defined by $\mathscr{N}(x) = x\sigma(x)$. It is easy to see that $\mathscr{N}(xy) = \mathscr{N}(x)\mathscr{N}(y)$, so that the elements of norm one in this algebra form a group, $\mathscr{G}(F; \alpha, \beta)$. By extending the field of scalars to the function field $F(t)$, and extending the involution σ by $t \to t^{-1}$, we obtain the algebra $\mathscr{H}(t)$. We denote the norm one group for this algebra by $\mathscr{G}(F(t); \alpha, \beta)$. There is a natural representation of the quaternion algebra to the matrix algebra of degree two over the field $F(\sqrt{\alpha})$ via

$$I \to \begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & -\sqrt{\alpha} \end{bmatrix} \quad \text{and} \quad J \to \begin{bmatrix} 0 & 1 \\ \beta & 0 \end{bmatrix}.$$

In this representation the element $x = a1 + bI + cJ + dK$ is represented by the matrix

$$\begin{bmatrix} a + b\sqrt{\alpha} & c + d\sqrt{\alpha} \\ \beta(c - d\sqrt{\alpha}) & a - b\sqrt{\alpha} \end{bmatrix}.$$

The norm is represented by the determinant, so that $\mathscr{N}(x) = r\sigma(r) - \beta s\sigma(s)$, where $r = a + b\sqrt{\alpha}$, $s = c + d\sqrt{\alpha}$, with $\sigma(\sqrt{\alpha}) = -\sqrt{\alpha}$. The representation is extended similarly to $\mathscr{H}(t)$.

In this article we shall consider the subgroups where the coefficient rational functions are in the ring of Laurant polynomials, $\mathscr{G}(F[t, t^{-1}]; \alpha, \beta)$, and its congruence subgroup Γ which is the kernel of the homomorphism

$$\mathscr{G}(F[t, t^{-1}]; \alpha, \beta) \to \mathscr{G}(F; \alpha, \beta)$$

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via $t \to 1$. In the representation above this means that r, s are Laurant polynomials over $F(\sqrt{\alpha})$, and $r(1) = 1$, $s(1) = 0$. Let $L = F(\sqrt{\alpha})$ and $A = L[t, t^{-1}]$.

Theorem 1. *Any subgroup of* Γ *that has no nontrivial unipotents is a free group.*

Proof. We consider the action of $\mathrm{SL}_2(L(t))$ on the Bruhat-Tits tree T_v for the discrete valuation on $L(t)$ at infinity, $v(f/g) = \mathrm{degree}(g) - \mathrm{degree}(f)$, for f and g in $L[t]$. The stabilizers of the vertices of T_v are conjugates of $\mathrm{SL}_2(B)$, where B is the valuation ring of v. It follows that the trace of an element of $\mathrm{SL}_2(L(t))$ that fixes a vertex is an element of B. Thus if an element of Γ fixes a vertex, its trace is in $A \cap B$. An element of $A \cap B$ has the form $t^{-n}f(t)$ where $f(t)$ is a polynomial from $L[t]$ and $\deg(f) \leq n$ and thus $A \cap B = L[t^{-1}]$. Thus the trace of an element $\gamma \in \Gamma$ that stabilizes a vertex of the tree is of the form $r + \sigma(r)$, where $r \in A$ and, therefore, $r + \sigma(r) \in L[t^{-1}]$ and $r(1) = 1$. Since this trace is invariant under σ, it must be constant, and hence by evaluation at $t = 1$, it follows immediately that γ is unipotent. We have thus shown that any subgroup of Γ without unipotents acts freely on this tree so by Serre's theorem [S, p. 27] it is a free group.

Theorem 2. *If* $\mathscr{H}$ *is a nonsplit quaternion algebra then* Γ *is a free group.*

Proof. As in the proof of Theorem 1, an element stabilizing a vertex of the tree yields elements r, s such that $r\sigma(r) - \beta s\sigma(s) = 1$, $r(1) = 1$, $s(1) = 0$, $\sigma(r) = 2 - r$. Hence $-\beta s\sigma(s) = (r-1)^2$ so that for $p = r - 1$, $\beta s\sigma(s) = p\sigma(p)$. If p is nonzero, we may multiply s, p by suitable powers of t so that they are polynomials in t, and then cancelling common powers of $t - 1$ from each side of this equation obtain $p(1)/s(1) \in L$ has norm β. This contradicts the fact that the algebra is nonsplit [L, p. 58]. Thus $r = 1$, so that any element of a stabilizer subgroup is trivial. Thus the group Γ acts freely on the tree.

Let F be a field of level $s(F) \geq 4$ [L, p. 302], that is, -1 is not a square or a sum of two squares of elements from F. In this case, the algebra with $\alpha = -1$, $\beta = -1$ is nonsplit. Let $L = F(i)$ be the extension of degree 2 over F, where $i^2 = -1$.

Corollary. *Let* $L = F(i)$, $s(F) \geq 4$. *Then the group* $\Gamma = \mathrm{SU}_2(L[t, t^{-1}], t-1)$ *is a free group.*

Proof. With involution σ so that $i \to -i$, $t \to t^{-1}$, the group of norm one elements Γ in this case is $\mathrm{SU}_2(L[t, t^{-1}], t-1)$, the kernel of the homomorphism

$$\mathrm{SU}_2(L[t, t^{-1}]) \to \mathrm{SU}_2(L),$$

where of course SU_2 means the group of 2 by 2 matrices X, so that $\sigma(X)^t = X^{-1}$. The result now follows immediately from Theorem 2 and the remarks above.

It would be interesting to know if Pollen's result that the matrices with linear entries form a set of free generators of this unitary group is also valid in this more general context.

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