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ON INDUCED REPRESENTATIONS OF DISCRETE GROUPS

MICHAEL W. BINDER

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ABSTRACT. The paper deals with induced representations $\operatorname{ind}_H^G \sigma$ of a locally compact group G where H is an open subgroup. Using "elementary intertwining operators", we first describe the commutant $\operatorname{ind}_H^G \sigma(G)'$ (also in the case of realizing the induced representations with positive definite measures). Then criteria for irreducibility and pairwise disjointness of induced representations are given. Finally, special attention is devoted to abelian subgroups H.

INTRODUCTION

Induced representations of locally compact groups have been studied by Mackey [7, 8], Blattner [1, 2], Kleppner [5], and many others. In particular, the case where the subgroup is open yields various results concerning, for example, the irreducibility and disjointness.

We will call an important tool "elementary intertwining operators". They have been used, for example, by Mackey [7], Corwin [3], and Kleppner [5, 6]. We briefly summarize the results in a slightly generalized version and give a criterion for the boundedness of these operators. We note that induced representations are irreducible or disjoint if there exists no nontrivial or no nonzero elementary intertwining operator, respectively. Thus we get another proof of the results due to Mackey.

Moreover, in many cases, the commutant is generated by the elementary intertwining operators. (This was claimed by Corwin [3], but his proof seems incomplete.) This result also gives a description of the commutant using the realization of induced representations with positive measures (introduced by Blattner [2]).

Inducing with finite-dimensional representations, we get an important simplification, which has been noted by Kleppner [5]. Besides Mackey's and Corwin's criteria for irreducibility [7, 3], this also yields a generalization of Obata's [10] criterion for pairwise disjointness. Examples will demonstrate its use.

Section 4 is devoted to representations that are induced from abelian subgroups. The main result gives a criterion for the existence of uncountably many

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pairwise disjoint, irreducible induced representations.

Notation. Let G be a locally compact group, H (resp. H_1, H_2) open subgroups of G, and σ (resp. σ_1, σ_2) continuous unitary representations of H (resp. H_1 and H_2) on a Hilbert space $\mathscr{H}^0$ (resp. $\mathscr{H}_1^0$, $\mathscr{H}_1^0$). $\langle G/H\rangle$ denotes a system of representatives of the cosets G/H.

$\Theta_g^2 := \langle H_2/H_2 \cap gH_1g^{-1}\rangle$ and $\Theta_{g^{-1}}^1 := \langle H_1/H_1 \cap g^{-1}H_2g\rangle$ for $g \in G$. If $H_1 = H_2 = H$ we simply write Θ_g or $\Theta_{g^{-1}}$.

As in [3], we define the subgroup

$$\mathscr{Q}_H := \{g \in G : [H : g^{-1}Hg \cap H] < \infty \text{ and } [H : gHg^{-1} \cap H] < \infty\}.$$

$\operatorname{ind}_H^G \sigma$ (or simply $\operatorname{ind} \sigma$) is the induced representation on the Hilbert space

$$\mathscr{H}_\sigma := \left\{ f\colon G \to \mathscr{H}^0 : f(gh) = \sigma(h^{-1})f(g) \text{ for every } g \in G,\ h \in H; \right.$$
$$\left. \sum_{g\in\langle G/H\rangle} \|f(g)\|^2 < \infty \right\}$$

defined by $\operatorname{ind}_H^G \sigma(g)f(y) := f(g^{-1}y)$ for $f \in \mathscr{H}_\sigma$, $y \in G$.

For $g \in G$ and $v \in \mathscr{H}^0$ the function $\delta_g^v \in \mathscr{H}_\sigma$ is defined by

$$\delta_g^v\colon G \to \mathscr{H}^0, \qquad y \mapsto \begin{cases} \sigma(y^{-1}g)v & \text{if } y \in gH, \\ 0 & \text{otherwise}. \end{cases}$$

For $g \in G$ the representation σ^g of $g^{-1}Hg$ is given by $\sigma^g(h) := \sigma(ghg^{-1})$ (for $h \in g^{-1}Hg$).

$\mathscr{R}_a(\operatorname{ind}_{H_1}^G \sigma_1, \operatorname{ind}_{H_2}^G \sigma_2)$ is the set of all closed operators A such that $\{\delta_g^v : g \in G,\ v \in \mathscr{H}_1^0\} \subseteq D(A)$, $\operatorname{ind}_{H_1}^G \sigma_1(g)D(A) \subseteq D(A)$, and $\operatorname{ind}_{H_2}^G \sigma_2(g) \cdot A = A \cdot \operatorname{ind}_{H_1}^G \sigma_1(g)|_{D(A)}$ for every $g \in G$, where $D(A)$ is the domain of A.

$\mathscr{L}(\mathscr{H}_1^0, \mathscr{H}_2^0)$ denotes the set of bounded operators from $\mathscr{H}_1^0$ to $\mathscr{H}_2^0$; and $\mathscr{L}(\mathscr{H}^0) := \mathscr{L}(\mathscr{H}^0, \mathscr{H}^0)$.

For any representation π of G, let $\pi(G)'$ denote the commutant of $\pi(G)$.

1. Elementary intertwining operators

Analogous to [7, 3], we define for $A \in \mathscr{R}_a(\operatorname{ind}_{H_1}^G \sigma_1, \operatorname{ind}_{H_2}^G \sigma_2)$ and $g \in G$ the operator $A_g \in \mathscr{L}(\mathscr{H}_1^0, \mathscr{H}_2^0)$ by $A_g\colon v \in \mathscr{H}_1^0 \mapsto A\delta_e^v(g^{-1}) \in \mathscr{H}_2^0$. As in [3] we see that A is determined by A_g, $g \in G$. Since $(A_g)^* = (A^*)_{g^{-1}}$ (cf. [3]), A_g is bounded by the theorem of Hellinger-Toeplitz.

Moreover, $A_{h_2gh_1} = \sigma_2(h_2)A_g\sigma_1(h_1)$ for $A \in \mathscr{R}_a(\operatorname{ind}_{H_1}^G \sigma_1, \operatorname{ind}_{H_2}^G \sigma_2)$ and $g \in G$, $h_1 \in H_1$, $h_2 \in H_2$ (cf. [3, Theorem 2]). So we define

Definition 1.1. $A \in \mathscr{R}_a(\operatorname{ind}_{H_1}^G \sigma_1, \operatorname{ind}_{H_2}^G \sigma_2)$ will be called an *elementary intertwining operator* $T(g, V)$ (where $g \in G$, $V \in \mathscr{L}(\mathscr{H}_1^0, \mathscr{H}_2^0)$) if

$$A_m = \begin{cases} \sigma_2(h_2)V\sigma_1(h_1) & \text{if } m = h_2gh_1 \in H_2gH_1, \\ 0 & \text{if } m \notin H_2gH_1. \end{cases}$$

It is useful to note the two formulas (for $f \in \mathscr{H}_{\sigma_1}$, $v \in \mathscr{H}_1^0$, and $y, m \in G$)

$$T(g, V)f(y) = \sum_{h \in \Theta_g^2} \sigma_2(h) V f(yhg),$$

$$T(g, V)\delta_m^v = \sum_{h \in \Theta_{g^{-1}}^1} \delta_{mhg^{-1}}^{V\sigma_1(h^{-1})v}.$$

We recall an important result due to Mackey [7]:

Proposition 1.2. (i) *For $g \in G$ and $V \in \mathscr{L}(\mathscr{H}_1^0, \mathscr{H}_2^0)$ there exists an elementary intertwining operator $T(g, V)$ if and only if the operator V satisfies*

$(\mathrm{t_g})$ $\qquad \sigma_2(ghg^{-1})V = V\sigma_1(h)$ *for every* $h \in H_1 \cap g^{-1}H_2 g$

and

$(\mathrm{s_g})$

$$\sum_{h \in \Theta_{g^{-1}}^1} \|V\sigma_1(h^{-1})v\|^2 < \infty \quad \textit{for every } v \in \mathscr{H}_1^0,$$

$$\sum_{h \in \Theta_g^2} \|V^*\sigma_2(h^{-1})w\|^2 < \infty \quad \textit{for every } w \in \mathscr{H}_2^0.$$

(ii) *Given $A \in \mathscr{R}_a(\operatorname{ind}_{H_1}^G \sigma_1, \operatorname{ind}_{H_2}^G \sigma_2)$ and $g \in G$, the operator A_g satisfies* $(\mathrm{t_g})$ *and* $(\mathrm{s_g})$.

Remark. Concerning $(\mathrm{s_g})$, it should be noted that for $V \in \mathscr{L}(\mathscr{H}_1^0, \mathscr{H}_2^0)$ and $(h_i)_{i \in I} \subset H_1$ the following conditions are equivalent:

(i) $\sum_{i \in I} \|V\sigma_1(h_i^{-1})v\|^2 < \infty$ for every $v \in \mathscr{H}_1^0$,

(ii) there is a constant $c > 0$ such that

$$\sum_{i \in I} \|V\sigma_1(h_i^{-1})v\|^2 \le (c \cdot \|v\|)^2 \quad \text{for every } v \in \mathscr{H}_1^0,$$

(iii) $\sum_{i \in I} \sigma_1(h_i)V^* w_i$ exists for every $(w_i)_{i \in I} \subset \mathscr{H}_2^0$ with $\sum_{i \in I} \|w_i\|^2 < \infty$.

To prove (i) $\Rightarrow$ (ii), consider $T\colon v \in \mathscr{H}_1^0 \mapsto (V\sigma_1(h_i^{-1})v)_{i \in I} \in l^2(I, \mathscr{H}_2^0)$, which is continuous since its graph is closed. Also the other implications are proved by common techniques of functional analysis.

Regarding the polar decomposition (resp. the spectral projections as in [11, 15.12]) we get the next proposition, which yields another proof of Mackey's criteria for irreducibility and disjointness.

Proposition 1.3. *If there is a nonzero operator $A \in \mathscr{R}_a(\operatorname{ind}_{H_1}^G \sigma_1, \operatorname{ind}_{H_2}^G \sigma_2)$ (resp. an operator $A \in \mathscr{R}_a(\operatorname{ind}_H^G \sigma, \operatorname{ind}_H^G \sigma) - \mathbb{C}\mathbf{1}$) then there is a nonzero bounded intertwining operator (resp. a nontrivial bounded operator in $\operatorname{ind} \sigma(G)'$). Thus $\operatorname{ind} \sigma_1$ and $\operatorname{ind} \sigma_2$ are disjoint (resp. $\operatorname{ind} \sigma$ is irreducible) if and only if there exist no nonzero elementary intertwining operators (resp. only the trivial elementary intertwining operators $T(e, \lambda\mathbf{1})$).*

Proposition 1.4. *Let $g \in G$ and assume $V \in \mathscr{L}(\mathscr{H}_1^0, \mathscr{H}_2^0)$ to satisfy conditions $(\mathrm{t_g})$ and $(\mathrm{s_g})$. If $|\Theta_{g^{-1}}^1| < \infty$ or $|\Theta_g^2| < \infty$, then the elementary intertwining operator $T(g, V)$ is continuous.*

Proof. Take $f \in D(T(g, V)) \subseteq \mathscr{H}_{\sigma_1}$ and $l \in \mathscr{H}_{\sigma_2}$, and suppose $|\Theta_{g^{-1}}^1| < \infty$. By $(\mathrm{s_g})$ and the remark to Proposition 1.2, there exists a constant c^* such that

$\sum_{h\in\Theta_g^2} \|V^*\sigma_2(h^{-1})l(k)\|^2 \le c^{*2}\|l(k)\|^2$. Using the inequality of Cauchy-Schwarz we get

$$\begin{aligned} |\langle T(g, V)f, l\rangle| &= \left|\sum_{k\in\langle G/H_2\rangle}\sum_{h\in\Theta_g^2}\langle\sigma_2(h)Vf(khg), l(k)\rangle\right| \\ &\le \sum_{k\in\langle G/H_2\rangle}\sum_{h\in\Theta_g^2}\|f(khg)\|\cdot\|V^*\sigma_2(h^{-1})l(k)\| \\ &\le \left(\sum_{k\in\langle G/H_2\rangle}\sum_{h\in\Theta_g^2}\|f(khg)\|^2\right)^{1/2} \\ &\quad\cdot\left(\sum_{k\in\langle G/H_2\rangle}\sum_{h\in\Theta_g^2}\|V^*\sigma_2(h^{-1})l(k)\|^2\right)^{1/2} \\ &\le |\Theta_{g^{-1}}^1|\cdot\|f\|\cdot\|l\|\cdot c^*. \end{aligned}$$

The last inequality holds because $\{khg\colon k\in\langle G/H_2\rangle,\ h\in\Theta_g^2\}$ meets every coset of G/H_1 in exactly $|\Theta_{g^{-1}}^1|$ elements. (Note that H_1g^{-1} meets exactly $|\Theta_{g^{-1}}^1|$ different cosets of G/H_2.) Thus $T(g, V)$ is continuous. If $|\Theta_g^2| < \infty$, we similarly find that $T(g^{-1}, V^*) = T(g, V)^*$ is continuous, hence so is $T(g, V)$. □

Remark. In the following cases, all elementary intertwining operators are continuous:

(i) $H_1 = H_2 = H$ is an invariant subset in G,
(ii) G/H_1 or G/H_2 is finite,
(iii) $\dim\mathscr{H}_1^0$ or $\dim\mathscr{H}_2^0$ is finite;

for then $|\Theta_g^2| < \infty$ or $|\Theta_{g^{-1}}^1| < \infty$ for every $g\in G$. Concerning (i), this is obvious; concerning (ii), we note again that H_1g^{-1} meets exactly $|\Theta_{g^{-1}}^1|$ different cosets of G/H_2, and concerning (iii), this will be shown in Lemma 3.1.

2. The commutant ind $\sigma(G)'$

Let $\mathscr{T}$ denote the linear space that is spanned by the elementary intertwining operators and $\mathscr{T}' := \{S\in\mathscr{L}(\mathscr{H}_\sigma)\colon ST = TS|_{D(A)}\}$. Define $\mathscr{V}_H := \{g\in G\colon$ there exists a nonzero $V\in\mathscr{L}(\mathscr{H}^0)$ that satisfies $(\mathrm{s_g})$ and $(\mathrm{t_g})\}$.

Lemma 2.1. *$\mathscr{T}$ is a *-algebra if $|\Theta_g| < \infty$ for every $g\in\mathscr{V}_H$.*

Proof. The elementary intertwining operators are continuous by Proposition 1.4. Since $(T(m, V))^* = T(m^{-1}, V^*)$, we get $\mathscr{T}^*\subseteq\mathscr{T}$, and since $|\Theta_g| < \infty$ for every $g\in G$, we find $\mathscr{T}\mathscr{T}\subseteq\mathscr{T}$. □

Theorem 2.2. $\mathscr{T}'' = \operatorname{ind}\sigma(G)'$. *In particular, if $|\Theta_g| < \infty$ for every $g\in\mathscr{V}_H$ (e.g., if σ is finite dimensional, see* 3.1) *then $\mathscr{T}$ is weakly dense in* $\operatorname{ind}\sigma(G)'$.

Proof. Since $\operatorname{ind}\sigma(G)\subseteq\mathscr{T}'$, we get one of the inclusions. Now take $A\in\operatorname{ind}\sigma(G)'$ and $S\in\mathscr{T}'$. We will show $AS = SA$, which yields the other inclusion.

As we have the formulas $(Af)(y) = \sum_{k \in G/H} A_k f(yk)$ and

$$\|A\delta_m^v\|^2 = \sum_{k \in G/H} \|A_{k^{-1}} v\|^2,$$

there is a net $(A^\iota)_{\iota \in \mathscr{D}} \subset \mathscr{T}$ such that $A^\iota f(y) \to Af(y)$ for every $f \in \mathscr{H}_\sigma$, $y \in G$ and $A^\iota \delta_m^v \to A\delta_m^v$ for every $m \in G$, $v \in \mathscr{H}^0$. Now take $k \in G$ and $v \in \mathscr{H}^0$. As $S \in \mathscr{T}'$ we have $SA^\iota \delta_k^v = A^\iota S\delta_k^v$. Since $A^\iota \delta_k^v$ converges to $A\delta_k^v$ and S is continuous, $SA^\iota \delta_k^v$ converges to $SA\delta_k^v$ in $\mathscr{H}_\sigma$. But on the other hand, $SA^\iota \delta_k^v = A^\iota S\delta_k^v$ converges pointwise to $AS\delta_k^v$. Therefore, $SA\delta_k^v = AS\delta_k^v$ (note $\|f_\iota(y) - f(y)\|^2 \le \|f_\iota - f\|^2$), whence $AS = SA$. Applying Proposition 1.4 and von Neumann's density theorem, we complete the proof. □

Induced representations realized with positive definite measures. Assume G to be discrete and σ to be a cyclic representation on $\mathscr{H}^0$ with cyclic vector v_0. We define

$$\varphi : g \in G \mapsto \begin{cases} \langle \sigma(h) v_0, v_0 \rangle & \text{if } g = h \in H, \\ 0 & \text{otherwise}. \end{cases}$$

$\|f\| := (\sum_{s \in G} (\sum_{g \in G} \overline{f(g)} f(gs)) \varphi(s))^{1/2}$ defines a seminorm on the space of functions with finite support. Let $\mathscr{H}_\varphi$ denote the corresponding Hilbert space. Then $\lambda_\varphi(g) f(x) := f(g^{-1}x)$ (for $f \in \mathscr{H}_\varphi$, $x, g \in G$) is a representation of G.

Blattner proved in [2] that $\operatorname{ind}_H^G \sigma$ and λ_φ are equivalent. (Remark. He used the formulas corresponding to the right invariant Haar measure.) For $k \in G$ let δ_k denote the element of $\mathscr{H}_\varphi$ associated to the function

$$g \in G \mapsto \begin{cases} 1 & \text{if } g = k, \\ 0 & \text{otherwise}. \end{cases}$$

Then a unitary operator intertwining $\operatorname{ind} \sigma$ and λ_φ is given by

$$U : \mathscr{H}_\sigma \to \mathscr{H}_\varphi, \qquad \delta_g^{\sigma(h)v_0} \mapsto U\delta_g^{\sigma(h)v_0} := \delta_{gh}.$$

If φ is a (normalized) character then the commutant $\lambda_\varphi(G)'$ is well known (see [13]): For $g \in G$ let $\varrho_\varphi(g)$ denote the operator $\varrho(g)f(x) := f(xg)$. Then $\lambda_\varphi(G)' = \varrho_\varphi(G)''$.

But in general, $\varrho_\varphi(g)$ is not well defined. Nevertheless we can prove

Theorem 2.3. *Let G be a discrete group and H a subgroup. If σ is a finite-dimensional, cyclic representation of H, then the set of all well-defined operators*

$$\sum_{i=1}^n \alpha_i \varrho_\varphi(g_i) f(x) := \sum_{i=1}^n \alpha_i f(xg_i), \qquad \alpha_i \in \mathbb{C},\ g_i, x \in G,\ n \in \mathbb{N},\ f \in \mathscr{H}_\varphi,$$

is dense in the commutant of $\lambda_\varphi(G)$.

Proof. Let $\sigma(h_1)v_0, \dots, \sigma(h_n)v_0$ be a base of $\mathscr{H}^0$. Take $A = T(g, V) \in \mathscr{T}$. For every $h \in H$ there are $\alpha_{h,i} \in \mathbb{C}$ such that

$$\sum_{i=1}^n \alpha_{h,i} \sigma(h_i) v_0 = V\sigma(h^{-1}) v_0.$$

We compute

$$A\delta_e^{v_0} = \sum_{h\in\Theta_{g-1}} \delta_{hg^{-1}}^{V\sigma(h^{-1})v_0} = \sum_{h\in\Theta_{g-1}} \sum_{i=1}^{n} \alpha_{h,i}\delta_{hg^{-1}h_i}^{v_0}.$$

Therefore

$$UAU^{-1}\delta_e = \sum_{h\in\Theta_{g-1}} \sum_{i=1}^{n} \alpha_{h,i}\varrho_\varphi(h_i^{-1}gh^{-1})\delta_e.$$

As δ_e is a cyclic vector for λ_φ we get

$$(*) \qquad UAU^{-1} = \sum_{h\in\Theta_{g-1}} \sum_{i=1}^{n} \alpha_{h,i}\varrho_\varphi(h_i^{-1}gh^{-1}).$$

Theorem 2.2 yields the assertion. □

Using equation $(*)$ as well as the results of the next section we also conclude

Corollary 2.4. *If σ is a one-dimensional representation of H then*

$$\lambda_\varphi(G)' = \left\{ \sum_{h\in\Theta_{g-1}} \varrho_\varphi(gh^{-1}) : g \in \mathscr{Q}_H \text{ and } \sigma^g|_{g^{-1}Hg\cap H} = \sigma|_{g^{-1}Hg\cap H} \right\}''.$$

If additionally $\mathscr{Q}_H$ is the normalizer $\mathscr{N}(H)$ of H then

$$\lambda_\varphi(G)' = \{\varrho_\varphi(g) : g \in \mathscr{N}(H) \text{ and } \sigma^g = \sigma\}''.$$

3. Inducing with finite-dimensional representations

A representation on a Hilbert space $\mathscr{H}^0$ is called finite dimensional if $\dim\mathscr{H}^0 < \infty$. In this case, the summability condition $(\mathrm{s_g})$ can be simplified.

Lemma 3.1. *Assume that $V \in \mathscr{L}(\mathscr{H}_1^0, \mathscr{H}_2^0)$ satisfies the summability condition $(\mathrm{s_g})$ for some $g \in G$.*

(i) *If $\dim\mathscr{H}_1^0 < \infty$ and $[H_1 : g^{-1}H_2g \cap H_1] = \infty$, then $V = 0$.*
(ii) *If $\dim\mathscr{H}_2^0 < \infty$ and $[H_2 : gH_1g^{-1} \cap H_2] = \infty$, then $V = 0$.*

Remark. If $\dim\mathscr{H}_1^0 = \dim\mathscr{H}_2^0 = 1$, this is obvious and noted by Mackey [7]. For arbitrary finite-dimensional irreducible representations σ_1, σ_2, Kleppner [5] and (for $\sigma_1 = \sigma_2$) Corwin [3] have given a proof using the compactness of the unit ball. We will present another short proof:

Proof. (i) Let $\{e_1, e_2, \ldots, e_n\}$ denote an orthonormal base of $\mathscr{H}_1^0$. Then it is well known that for every bounded operator $U \in \mathscr{L}(\mathscr{H}_1^0, \mathscr{H}_2^0)$ the inequality

$$\|U\| \le |||U||| = \left(\sum_{i=1}^{n} \|Ue_i\|^2\right)^{1/2}$$

holds, where $|||U|||$ denotes the Hilbert-Schmidt norm of U.

In view of $\|V\sigma_1(h^{-1})\| = \|V\|$, we get from $(\mathrm{s_g})$

$$\sum_{h\in\Theta^1_{g-1}} \|V\|^2 \le \sum_{h\in\Theta^1_{g-1}} \sum_{i=1}^{n} \|V\sigma_1(h^{-1})e_i\|^2 = \sum_{i=1}^{n} \sum_{h\in\Theta^1_{g-1}} \|V\sigma_1(h^{-1})e_i\|^2 < \infty.$$

Thus $V = 0$ or $|\Theta^1_{g^{-1}}|$ is finite and (i) is proved; (ii) can be proved similarly. □

Thus, by Propositions 1.2 and 1.3 we get the following theorem, which can be obtained from Kleppner's paper [5] and generalizes results due to Obata [10] and Corwin [3] (published later).

Theorem 3.2. *Let σ (resp. σ_1, σ_2) be a representation of the open subgroup H (resp. H_1 and H_2).*

(i) [3, Corollary 1] *Assume σ is finite dimensional. Then the induced representation* ind σ *is irreducible if and only if $\sigma^g|_{g^{-1}Hg\cap H}$ and $\sigma|_{g^{-1}Hg\cap H}$ are disjoint for every $g \in \mathscr{Q}_H - H$.*

(ii) *Assume σ_1 and σ_2 are finite dimensional. Then* ind σ_1 *and* ind σ_2 *are disjoint if and only if $\sigma_2^g|_{g^{-1}H_2g\cap H_1}$ and $\sigma_1|_{g^{-1}H_2g\cap H_1}$ are disjoint for every $g \in G$ such that $[H_1 : g^{-1}H_2g \cap H_1] < \infty$ and $[H_2 : gH_1g^{-1} \cap H_2] < \infty$.*

(iii) *Assume σ_1 is finite dimensional (and σ_2 of arbitrary dimension). Then* ind σ_1 *and* ind σ_2 *are disjoint if $\sigma_2^g|_{g^{-1}H_2g\cap H_1}$ and $\sigma_1|_{g^{-1}H_2g\cap H_1}$ are disjoint for every $g \in G$ such that $[H_1 : g^{-1}H_2g \cap H_1] < \infty$.*

In each case it is sufficient to consider the elements g of a system of representatives of the double cosets $H \setminus G/H$ or $H_2 \setminus G/H_1$, respectively.

We will need the above generalization to discuss the following

Examples. (1) Let G be the Heisenberg group over $\mathbb{Z}$. A realization of G is given by $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ endowed with $(a, b, c)(a', b', c') = (a + a', b + b', c + c' + ab')$. Let $H := \{0\} \times \mathbb{Z} \times \mathbb{Z} < G$. Obviously it is abelian, the dual group $\widehat{H}$ is topologically isomorphic to $\mathbf{T}\times\mathbf{T}$ (where $\mathbf{T} := \{\exp(2\pi i\alpha) \in \mathbb{C} : 0 \leq \alpha < 1\}$). From (ii) (and (i)) we conclude that $\operatorname{ind}_H^G(\exp(2\pi i\beta_1)\times\exp(2\pi i\gamma_1))$ and $\operatorname{ind}_H^G(\exp(2\pi i\beta_2)\times\exp(2\pi i\gamma_2))$ are disjoint if and only if $(\beta_2-\beta_1+\gamma_1\mathbb{Z})\cap\mathbb{Z} = \varnothing$ or $\gamma_1 \neq \gamma_2$ (and that $\operatorname{ind}_H^G(\exp(2\pi i\beta)\times\exp(2\pi i\gamma))$ is irreducible iff $\gamma \notin \mathbb{Q}$).

(2) Let $S_{\mathbb{N}}$ be the infinite symmetric group. Choose an $n \in \mathbb{N}$ and define $H := \{f \in S_{\mathbb{N}} : f(\{1, \ldots, n\}) = \{1, \ldots, n\}\}$. Let σ_1 be a finite-dimensional and σ_2 an arbitrary representation of H. As $[H : gHg^{-1}\cap H] < \infty$ iff $g \in H$, the induced representations are disjoint if (and only if) σ_1 and σ_2 are disjoint.

(3) Let $G = A \circ_s N$ be a semidirect product of two infinite discrete groups, and let σ and τ be representations of A and N, respectively. If σ or τ is finite dimensional then $\operatorname{ind}_A^G \sigma$ and $\operatorname{ind}_N^G \tau$ are disjoint.

4. Inducing from an abelian subgroup

In this section we assume G to be a discrete group. For an abelian subgroup H of G, $\widehat{H}$ denotes the (compact) dual group (i.e., the set of irreducible, one-dimensional representations of H), μ the Haar measure of $\widehat{H}$. Define for $g \in G$ the following subgroup of H:

$$K_g := \{h^{-1}ghg^{-1} : h \in g^{-1}Hg \cap H\}.$$

Let $\mathscr{C}(g)$ be the centralizer of g in G. We note that $[H : \mathscr{C}(g) \cap H] = |\{h^{-1}gh : h \in H\}|$ and that $[H : \mathscr{C}(g) \cap H] = \infty$ for every $g \in G - H$ if and only if $|K_g| = \infty$ for every $g \in \mathscr{Q}_H - H$.

Theorem 3.2 yields

Theorem 4.1. *Either* $K_g = \{e\}$ *for some* $g \in \mathscr{Q}_H - H$ *and hence* $\operatorname{ind}\sigma$ *is reducible for all* $\sigma \in \widehat{H}$, *or* $\operatorname{ind}\sigma$ *is irreducible for every faithful character* $\sigma \in \widehat{H}$. *In particular, if* $[H : \mathscr{C}(g) \cap H] = \infty$ *for every* $g \in G - H$, *then* $\operatorname{ind}\sigma$ *is irreducible for every faithful character* $\sigma \in \widehat{H}$.

Lemma 4.2. *Let* G *be discrete.*

(i) *Take* $g \in G$. *Then* $|K_g| = \infty$ *if and only if* $\{\sigma \in \widehat{H} : \sigma|_{K_g} = \iota\}$ *is a set of measure zero, where* ι *is the unit representation of* K_g.

(ii) *Take* $\sigma_0 \in \widehat{H}$ *and* $g \in \mathscr{Q}_H$. *If* $|H| = \infty$ *then* $\{\sigma \in \widehat{H} : \sigma_0|_{g^{-1}Hg\cap H} = \sigma|_{g^{-1}Hg\cap H}\}$ *is a set of measure zero.*

Proof. For $\varrho \in \widehat{K}_g$ define $\widehat{H}_{g,\varrho} := \{\sigma \in \widehat{H} : \sigma|_{K_g} = \varrho\}$. By [4, 24.12], $\widehat{H}_{g,\varrho} \neq \varnothing$, and for $\sigma \in \widehat{H}_{g,\varrho}$ we have $\widehat{H}_{g,\varrho} = \sigma\widehat{H}_{g,\iota}$. Hence $\mu(\widehat{H}_{g,\varrho}) = \mu(\widehat{H}_{g,\iota})$. But $\widehat{H} = \bigcup_{\varrho\in\widehat{K}_g} \widehat{H}_{g,\varrho}$, and as $|K_g| = \infty$ iff $|\widehat{K}_g| = \infty$ and as $0 < \mu(\widehat{H}) < \infty$, we get (i). (ii) is proved with the same arguments. □

Together with Theorem 3.2 we conclude

Theorem 4.3. *Let* G *be a discrete group,* H *an abelian subgroup, and assume* $\mathscr{Q}_H/H$ *to be countable.*

(i) $\operatorname{ind}_H^G\sigma$ *is irreducible for* μ*-almost every* $\sigma \in \widehat{H}$ *if and only if* $[H : \mathscr{C}(g) \cap H] = \infty$ *for every* $g \in G - H$.

(ii) *Let* $\sigma_0 \in \widehat{H}$. *If* $|H| = \infty$ *then* $\operatorname{ind}\sigma$ *and* $\operatorname{ind}\sigma_0$ *are disjoint for* μ*-almost every* $\sigma \in \widehat{H}$. *In particular, there are uncountably many pairwise disjoint induced representations* $\operatorname{ind}\sigma$.

Remarks. (1) Even if $[H : \mathscr{C}(g) \cap H] = \infty$ for every $g \in G - H$ there may exist characters $\sigma \in \widehat{H}$ such that $\operatorname{ind}\sigma$ is reducible (consider $\operatorname{ind}_N^G 1$ in the following example (1)).

(2) If $\mathscr{Q}_H/H$ is uncountable, this theorem fails in general.

Corollary 4.4. *If* H *is torsionfree, then either* $K_g = \{e\}$ *for some* $g \in \mathscr{Q}_H - H$ *and* $\operatorname{ind}\sigma$ *is reducible for every* $\sigma \in \widehat{H}$, *or* $\operatorname{ind}\sigma$ *is irreducible for* μ*-almost every* $\sigma \in \widehat{H}$.

Examples. (1) Let $G = A \circ_s N$ be a semidirect product of an abelian invariant group N and a countable group A (each endowed with the discrete topology). Then the following conditions are equivalent:

(i) $|\{n^{-1}\varphi(a)n : n \in N\}| = \infty$ for every $a \in A$, $a \neq e$,

(ii) $\operatorname{ind}_N^G\sigma$ is irreducible for μ-almost every $\sigma \in \widehat{N}$.

(2) Let $G = \mathrm{SL}(2, \mathbb{Z})$ be the special linear group over $\mathbb{Z}$ and put

$$H := \left\{ \begin{pmatrix} \varepsilon & a \\ 0 & \varepsilon \end{pmatrix} : \varepsilon = \pm 1,\ a \in \mathbb{Z} \right\}.$$

Then for μ-almost every $\sigma \in \widehat{H}$, and in particular for every faithful character σ, the induced representation $\operatorname{ind}\sigma$ is irreducible.

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MATHEMATISCHES INSTITUT, TECHNISCHE UNIVERSITÄT MÜNCHEN, ARCISSTR. 21, D-W8000 MÜNCHEN 2, GERMANY