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A STRENGTHENING OF LETH AND MALITZ'S UNIQUENESS CONDITION FOR SEQUENCES

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ABSTRACT. A series $\sum a_n$ of nonnegative real numbers is determined up to a constant multiple by the comparisons of its subsums, provided that $a_i \le \sum_{i>n} a_i$ and $\{a_n\}$ decreases to 0. This characterization is an improvement of Leth and Malitz's results.

1. INTRODUCTION

Any sequence $\{a_n\}$ of nonnegative numbers induces a preordering on subsets of N in the following way

$$I \preceq J \quad \text{if and only if} \quad \sum_{i\in I} a_i \preceq \sum_{j\in J} a_j.$$

Leth [4] gave conditions on the sequence $\{a_n\}$ under which the induced preordering determines the sequence up to a constant multiple. He proved

Theorem 1. *Let $\{a_n\}$ and $\{b_n\}$ be two nonincreasing sequences of real numbers such that*

(i) $a_n > 0$, $b_n > 0$, $\lim_{n\to\infty} a_n = 0$, *and* $\lim_{n\to\infty} b_n = 0$,
(ii) $a_n \le r_n = \sum_{i>n} a_i$ *and* $b_n \le R_n = \sum_{k>n} b_k$,
(iii) $\sum_{i\in I} a_i \le \sum_{j\in J} a_j$ *if and only if* $\sum_{i\in I} b_i \le \sum_{j\in J} b_j$ *for any* $I, J \subset N$.

Then there is a constant α such that $b_n = \alpha a_n$ for all $n \in N$.

Note that the condition $a_n \le r_n$ in (ii) is satisfied if and only if the set of subsums $E = \{\sum \varepsilon_n a_n\,;\ \varepsilon_n = 0, 1\}$ is an interval. For more on the structure of the set E one can consult [3].

This theorem can be seen as a result on purely atomic measures. In fact Chuaqui and Malitz [2] originated this problem by looking for necessary and sufficient conditions for the existence of σ-additive probability measures compatible with given preorderings.

Malitz [5] has strengthened Leth's result by proving the same conclusion under the weaker assumption

(iii)′ $\sum_{i\in I} a_i = \sum_{j\in J} a_j$ if and only if $\sum_{i\in I} b_i = \sum_{j\in J} b_j$ for all $I, J \subset N$.

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Recently Nymann [6] extended Leth and Malitz's results when the series $\sum a_n$ and $\sum b_n$ are convergent.

The main result of this work is to prove the conclusion of Theorem 1 under the weakest assumption

(iii)″ $\sum_I a_i = \sum_J a_j$ implies $\sum_I b_i = \sum_J b_j$ for all $I, J \subset N$.

2. Preliminaries and main result

The following result can be found in [4, 6].

Proposition 1. *Let $\{a_n\}$ be a sequence of real numbers. Assume that $0 \le a_{n+1} \le a_n \le r_n = \sum_{i>n} a_i$ for all $n \in N$ and $\lim_{n\to\infty} a_n = 0$. Then*

1. *For every $0 \le r \le \sum_{i=1}^{\infty} a_i$, there exists $K \subset N$ such that $r = \sum_{i\in K} a_i$. If $r < \sum_{i=1}^{\infty} a_i$ then one can assume that $N - K$ is infinite. If $0 < r$ then one can assume K is infinite.*
2. *There exists $J_n \subset (n, \infty)$ such that $a_n = \sum_{i\in J_n} a_i$, with $\min J_n = n+1$ for all $n \in N$.*
3. *Assume that the series $\sum_n a_n$ is divergent. Then if $\sum_I a_i < \sum_J a_j$, there exists $K \subset N - I$ such that $\sum_I a_i + \sum_K a_k = \sum_J a_j$.*

Let us remark that under the assumptions of Proposition 1 if $a_n = 0$ for some n, then $a_n = 0$ for every n. Therefore we will always assume that $a_n > 0$ for every n.

Definition 1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers. We will write $\{b_n\} \ll \{a_n\}$ if and only if $\sum_I a_i = \sum_J a_j$ implies $\sum_I b_i = \sum_J b_j$ for all $I, J \subset N$.

The proof of the following lemma can be found in [6].

Lemma 2. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers. Assume that $a_{n+1} \le a_n \le r_n = \sum_{i>n} a_i$ for all $n \in N$. Then $b_{n+1} \le b_n \le \sum_{i>n} b_i$ for all n, provided that $\{b_n\} \ll \{a_n\}$. Moreover if $\lim_{n\to\infty} a_n = 0$ then $\lim_{n\to\infty} b_n = 0$.*

The proof of the last statement is not given in [6], but it is not hard to deduce it. Indeed, since $\{b_n\}$ is decreasing, $\lim_{n\to\infty} b_n$ exists. Using Proposition 1, one can get $b_1 = \sum_{J_1} b_j$ where J_1 is an infinite subset of N. Therefore a subsequence of $\{b_n\}$ converges to 0, which implies that $\lim b_n = 0$.

Let us remark that if $\{b_n\} \ll \{a_n\}$ then it is not difficult to see that if $\sum a_n$ is divergent then $\sum b_n$ is also divergent. In the next proposition we prove the converse.

Proposition 2. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers. Assume that $a_{n+1} \le a_n \le r_n = \sum_{i>n} a_i$ and $\sum a_n$ is convergent. Then if $\{b_n\} \ll \{a_n\}$, $\sum b_n$ is convergent.*

Proof. Assume to the contrary that $\sum b_n$ is divergent and $\{b_n\} \ll \{a_n\}$. Define f and $\bar{f}$ by

$$f\left(\sum_{i\in I} a_i\right) = \sum_{i\in I} b_i$$

and

$$\bar{f}\left(\sum_{i\in I} b_i\right) = \sup\left\{\sum_{j\in J} a_j\,;\ \sum_{j\in J} b_j = \sum_{i\in I} b_i\right\}.$$

Our assumptions on $\{a_n\}$ and $\{b_n\}$ imply that f is defined on $[0, \sum a_i]$ (with values in $[0, \infty]$) and $\bar{f}$ is defined on $[0, \infty]$ (with values in $[0, \sum a_n]$). First let us prove that

$$f(\bar{f}(x)) = x \quad \text{for every } x \in [0, \infty).$$

Indeed, let $x \in (0, \infty)$. Then by Proposition 1, the assumptions on $\{a_n\}$ imply that $\bar{f}(x) = \sum_{i\in M} a_i$ where M is a subset of N. Also if $x = \sum_I b_i$ then $\bar{f}(x) = \sum_M a_i \geq \sum_I a_i$. So we can assume that M is infinite. Set $\delta_n = \sum_{M_n} a_i$ where $M_n = M \cap (n, \infty)$ for all $n \in N$. Then by definition of $\bar{f}$ one can find $L \subset N$ such that

$$\sum_{i\in M} a_i - \delta_n \leq \sum_{i\in L} a_i \leq \sum_{i\in M} a_i \quad \text{and} \quad \sum_{i\in L} b_i = x.$$

Clearly we have $\sum_L a_i - \sum_{\{i\in M;\ i\leq n\}} a_i \leq \sum_{i>n} a_i$. Then by Proposition 1, we obtain

$$\sum_{i\in L} a_i - \sum_{\{i\in M;\ i\leq n\}} a_i = \sum_{i\in J_n} a_i$$

for some $J_n \subset N - \{0, 1, \ldots, n\}$. Since $\{b_n\} \ll \{a_n\}$, we have

$$\sum_{i\in L} b_i = \sum_{\{i\in M;\ i\leq n\}} b_i + \sum_{i\in J_n} b_i = x.$$

Then $\sum_{\{i\in M;\ i\leq n\}} b_i \leq x$ for all $n \in N$. So $\sum_{i\in M} b_i \leq x$ holds.

Assume that $\sum_{i\in M} b_i < x$. Then the third conclusion of Proposition 1 implies that there exists K, a nonempty subset of $N - M$, such that $\sum_M b_i + \sum_K b_i = x$. So by the definition of $\bar{f}$ we have

$$\bar{f}(x) \geq \sum_{i\in M} a_i + \sum_{i\in K} a_i,$$

which contradicts the fact that $\bar{f}(x) = \sum_M a_i$. Therefore,

$$f(\bar{f}(x)) = x \quad \text{for all } x \in [0, \infty).$$

Next we prove that $\bar{f}$ is strictly increasing. Indeed, let $x_1, x_2 \in [0, \infty)$ with $x_1 < x_2$. Proposition 1(1) implies that $\bar{f}(x_1) \neq \bar{f}(x_2)$. Put $\bar{f}(x_1) = \sum_{M_1} a_i$ and $\bar{f}(x_2) = \sum_{M_2} a_i$. Then by using Proposition 1, one can find K, a nonempty subset of $N - M_2$, such that

$$\sum_{i\in M_1} b_i + \sum_{i\in K} b_i = \sum_{i\in M_2} b_i,$$

since $x_1 = f(\bar{f}(x_1)) = \sum_{M_1} b_i$ and $x_2 = f(\bar{f}(x_2)) = \sum_{M_2} b_i$. By the definition of $\bar{f}$ we get

$$\sum_{i\in M_1} a_i + \sum_{i\in K} a_i \leq \bar{f}(x_2) = \sum_{i\in M_2} a_i,$$

which implies that $\bar{f}(x_1) = \sum_{M_1} a_i < \bar{f}(x_2)$.

The last step consists of proving that $\bar{f}$ is continuous. In order to show that $\bar{f}$ is left-continuous (resp. right-continuous) at $x \in (0, \infty)$, it is enough to prove that for some $x_n < x$ (resp. $x_n > x$) with $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} \bar{f}(x_n) = \bar{f}(x)$ because $\bar{f}$ is increasing.

Left-continuity is easy to show. Indeed, let $x \in (0, \infty)$. Then $\bar{f}(x) = \sum_M a_i$ where M is an infinite subset of N. Since $f(\bar{f}(x)) = x$, we have $x = \sum_M b_i$. Set $\delta_n = \sum_{i \in M_n} b_i$ and $x_n = x - \delta_n$ where $M_n = \{i \in M;\ i > n\}$ for all $n \in N$. Then clearly we have $\lim_{n\to\infty} x_n = x$ and $\lim \bar{f}(x_n) = \bar{f}(x)$, since

$$\sum_{\{i \in M;\ i \le n\}} a_i \le \bar{f}(x_n) < \bar{f}(x).$$

So $\bar{f}$ is left-continuous on $(0, \infty)$.

To complete the proof of continuity of $\bar{f}$, let us show that $\bar{f}$ is right-continuous.

Let $x \in (0, \infty)$, and again set $\bar{f}(x) = \sum_M a_i$. Since $x = \sum_M b_i$, one can assume that $N - M$ is infinite. Put $N - M = \{k_1, k_2, \dots\}$ and $x_n = x + b_{k_n}$ for all $n \in N$. Then $\lim_{n\to\infty} x_n = x$. Since $\{x_n\}$ is decreasing, $\{\bar{f}(x_n)\}$ is a decreasing sequence with $\bar{f}(x_n) \ge \bar{f}(x)$ for all n. Then $\lim_{n\to\infty} \bar{f}(x_n) = \omega$ exists and $\omega \ge \bar{f}(x)$. Our assumptions on $\{a_n\}$ imply that $\omega = \sum_I a_i$ for some subset $I \subset N$. Since $\omega > 0$, one can assume that $N - I$ is infinite. Set

$$\omega_{n_0} = \omega + \sum_{\{i \in N - I;\ i > n_0\}} a_i$$

for $n_0 \in N$. Then one can find $l_0 \in N$ such that $\bar{f}(x_n) \le \omega_{n_0}$ for all $n \ge l_0$. Since $\omega \le \bar{f}(x_n)$ for all n, we get that

$$\bar{f}(x_n) = \sum_{\{i \in I;\ i \le n_0\}} a_i + \sum_{i \in J_n} a_i$$

for some $J_n \subset N - \{0, 1, 2, \dots, n_0\}$ for all $n \ge l_0$. Then

$$x_n = f(\bar{f}(x_n)) = \sum_{\{i \in I;\ i \le n\}} b_i + \sum_{i \in J_n} b_i$$

for all $n \le l_0$. In particular, $\sum_{\{i \in I;\ i \le n_0\}} b_i \le x_n$ holds for all $n \ge l_0$, which implies

$$x = \lim x_n \ge \sum_{\{i \in I;\ i \le n_0\}} b_i.$$

Since this is true for all $n_0 \in N$, we get $\sum_I b_i \le x$. And because

$$\omega = \sum_{i \in I} a_i \le \bar{f}\left(\sum_{i \in I} b_i\right) \le \bar{f}(x) \le \sum_{i \in I} a_i = \omega,$$

we get $\bar{f}(x) = \omega = \lim \bar{f}(x_n)$. So the proof of the continuity of $\bar{f}$ is complete.

Therefore $\bar{f}((0, \infty))$ is an interval. It is not hard to see that $\bar{f}((0, \infty)) = (0, \sum a_i)$. So there exists $x \in (0, \infty)$ such that $\bar{f}(x) = \sum_{i>n} a_i$ for $n > 1$, which implies that $x = \sum_{i>n} b_i$. This yields a contradiction with $\sum_{i>n} b_i = \infty$ for all $n \in N$. So the proof of Proposition 2 is complete.

The next theorem states the main result of this work.

Theorem 2. *Let $\{a_n\}$ be a sequence of real numbers such that $0 < a_{n+1} \leq a_n \leq \sum_{i>n} a_i$ for all n, and let $\lim_{n\to\infty} a_n = 0$. Let $\{b_n\}$ be a sequence of nonnegative real numbers such that $\{b_n\} \ll \{a_n\}$. Then there exists $\alpha \in R$ such that $b_n = \alpha a_n$ for all $n \in N$.*

Proof. Consider the function f defined on $[0, \sum_{i=1}^{\infty} a_i]$ by $f(\sum_I a_i) = \sum_I b_i$. If $\sum a_n$ is divergent then f is clearly increasing (by Proposition 1). And if $\sum a_n$ is convergent then $\sum b_n$ is convergent and again f is increasing (see [6]). Therefore, f is almost everywhere differentiable (see [7, p. 96]). Let f be differentiable at $x \in (0, \infty)$. Set $x = \sum_I a_i$ with $I \subset N$. Define

$$h_n = \begin{cases} a_n & \text{if } n \in N - I, \\ -a_n & \text{if } n \in I. \end{cases}$$

Then

$$\frac{f(x+h_n) - f(x)}{h_n} = \frac{b_n}{a_n}$$

for all $n \in N$. Since f is differentiable at x, we deduce that $\lim_{n\to\infty}(b_n/a_n) = \alpha$ exists. Assume that there exists n_0 such that $b_{n_0}/a_{n_0} \neq \alpha$. Put

$$A = \left\{n \in N;\ \frac{b_n}{a_n} \geq \frac{b_{n_0}}{a_{n_0}}\right\} \quad \text{if } \frac{b_{n_0}}{a_{n_0}} > \alpha.$$

Then A is a finite set. Therefore, using Proposition 1, there exists an infinite subset I of N such that $\sum_{i\in A} a_i = \sum_{i\in I} a_i$. So

$$\sum_{i\in A-I} a_i = \sum_{i\in I-A} a_i,$$

and since $\{b_n\} \ll \{a_n\}$, we have

$$\sum_{i\in A-I} b_i = \sum_{i\in I-A} b_i.$$

This yields, by the definition of A,

$$\frac{b_{n_0}}{a_{n_0}} \sum_{i\in A-I} a_i \leq \sum_{i\in A-I} b_i = \sum_{i\in I-A} b_i < \frac{b_{n_0}}{a_{n_0}} \sum_{i\in I-A} a_i.$$

Therefore, $\sum_{A-I} a_i = \sum_{I-A} a_i = 0$, which implies that $A = I$, contradicting the fact that A is finite and I is infinite. So $b_n = \alpha a_n$ for all n.

We complete the proof by noticing that if $b_{n_0}/a_{n_0} < \alpha$ then one can set

$$A = \left\{n \in N;\ \frac{b_n}{a_n} \leq \frac{b_{n_0}}{a_{n_0}}\right\}.$$

Theorem 2 can be interpreted as a result on purely atomic measures. For the nonatomic case, one can consult [1, 8]. In the next theorem an extension to arbitrary σ-finite measures is discussed. Notice that the finite case is proved in [6].

Theorem 3. *Let μ be a σ-finite measure. Assume that the range of μ is an interval. If μ is a purely atomic measure, we will assume that the μ-measure of the atoms decreases to 0. Then any measure μ' such that*

$$\mu(A) = \mu(B) \quad \textit{implies} \quad \mu'(A) = \mu'(B)$$

is proportional to μ, i.e., there exists $\alpha \in R$ such that $\mu' = \alpha\mu$.

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