A STRENGTHENING OF LETH AND MALITZ'S UNIQUENESS CONDITION FOR SEQUENCES

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Abstract. A series \( \sum a_n \) of nonnegative real numbers is determined up to a constant multiple by the comparisons of its subsums, provided that \( a_i \leq \sum_{i>n} a_i \) and \( \{a_n\} \) decreases to 0. This characterization is an improvement of Leth and Malitz's results.

1. Introduction

Any sequence \( \{a_n\} \) of nonnegative numbers induces a preordering on subsets of \( \mathbb{N} \) in the following way

\[ I \preceq J \text{ if and only if } \sum_{i \in I} a_i \leq \sum_{j \in J} a_j. \]

Leth [4] gave conditions on the sequence \( \{a_n\} \) under which the induced preordering determines the sequence up to a constant multiple. He proved

Theorem 1. Let \( \{a_n\} \) and \( \{b_n\} \) be two nonincreasing sequences of real numbers such that

(i) \( a_n > 0, \ b_n > 0, \ \lim_{n \to \infty} a_n = 0, \) and \( \lim_{n \to \infty} b_n = 0, \)

(ii) \( a_n \leq r_n = \sum_{i>n} a_i \) and \( b_n \leq R_n = \sum_{k>n} b_k, \)

(iii) \( \sum_{i \in I} a_i \leq \sum_{j \in J} a_j \) if and only if \( \sum_{i \in I} b_i \leq \sum_{j \in J} b_j \) for any \( I, J \subseteq \mathbb{N}. \)

Then there is a constant \( \alpha \) such that \( b_n = \alpha a_n \) for all \( n \in \mathbb{N}. \)

Note that the condition \( a_n \leq r_n \) in (ii) is satisfied if and only if the set of subsums \( E = \{ \sum \varepsilon_n a_n : \varepsilon_n = 0, 1 \} \) is an interval. For more on the structure of the set \( E \) one can consult [3].

This theorem can be seen as a result on purely atomic measures. In fact Chuaqui and Malitz [2] originated this problem by looking for necessary and sufficient conditions for the existence of \( \sigma \)-additive probability measures compatible with given preorderings.

Malitz [5] has strengthened Leth's result by proving the same conclusion under the weaker assumption

(iii)' \( \sum_{i \in I} a_i = \sum_{j \in J} a_j \) if and only if \( \sum_{i \in I} b_i = \sum_{j \in J} b_j \) for all \( I, J \subseteq \mathbb{N}. \)
Recently Nymann [6] extended Leth and Malitz's results when the series \( \sum a_n \) and \( \sum b_n \) are convergent.

The main result of this work is to prove the conclusion of Theorem 1 under the weakest assumption

(iii)'' \( \sum_I a_i = \sum_J a_j \) implies \( \sum_I b_i = \sum_J b_j \) for all \( I, J \subset N \).

2. Preliminaries and main result

The following result can be found in [4, 6].

**Proposition 1.** Let \( \{a_n\} \) be a sequence of real numbers. Assume that \( 0 < a_{n+1} \leq a_n \leq r_n = \sum_{i>n} a_i \) for all \( n \in N \) and \( \lim_{n \to \infty} a_n = 0 \). Then

1. For every \( 0 < r \leq \sum_{i=1}^{\infty} a_i \), there exists \( K \subset N \) such that \( r = \sum_{i \in K} a_i \). If \( r < \sum_{i=1}^{\infty} a_i \) then one can assume that \( N - K \) is infinite. If \( r > r \) then one can assume \( K \) is finite.
2. There exists \( J_n \subset (n, \infty) \) such that \( a_n = \sum_{i \in J_n} a_i \), with \( \min J_n = n + 1 \) for all \( n \in N \).
3. Assume that the series \( \sum a_n \) is divergent. Then if \( \sum_I a_i < \sum_J a_j \), there exists \( K \subset N - I \) such that \( \sum_I a_i + \sum_K a_k = \sum_J b_j \).

Let us remark that under the assumptions of Proposition 1 if \( a_n = 0 \) for some \( n \), then \( a_n = 0 \) for every \( n \). Therefore we will always assume that \( a_n > 0 \) for every \( n \).

**Definition 1.** Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of nonnegative real numbers. We will write \( \{b_n\} \ll \{a_n\} \) if and only if \( \sum_I b_i = \sum_J b_i \) implies \( \sum_I a_i = \sum_J a_j \) for all \( I, J \subset N \).

The proof of the following lemma can be found in [6].

**Lemma 2.** Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of nonnegative real numbers. Assume that \( a_{n+1} \leq a_n \leq r_n = \sum_{i>n} a_i \) for all \( n \in N \). Then \( b_{n+1} \leq b_n \leq \sum_{i>n} b_i \) for all \( n \), provided that \( \{b_n\} \ll \{a_n\} \). Moreover if \( \lim_{n \to \infty} a_n = 0 \) then \( \lim_{n \to \infty} b_n = 0 \).

The proof of the last statement is not given in [6], but it is not hard to deduce it. Indeed, since \( \{b_n\} \) is decreasing, \( \lim_{n \to \infty} b_n \) exists. Using Proposition 1, one can get \( b_1 = \sum_{j \in J_1} b_j \) where \( J_1 \) is an infinite subset of \( N \). Therefore a subsequence of \( \{b_n\} \) converges to 0, which implies that \( \lim b_n = 0 \).

Let us remark that if \( \{b_n\} \ll \{a_n\} \) then it is not difficult to see that if \( \sum a_n \) is divergent then \( \sum b_n \) is also divergent. In the next proposition we prove the converse.

**Proposition 2.** Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of nonnegative real numbers. Assume that \( a_{n+1} \leq a_n \leq r_n = \sum_{i>n} a_i \) and \( \sum a_n \) is convergent. Then if \( \{b_n\} \ll \{a_n\}, \sum b_n \) is convergent.

**Proof.** Assume to the contrary that \( \sum b_n \) is divergent and \( \{b_n\} \ll \{a_n\} \). Define \( f \) and \( \tilde{f} \) by

\[
 f\left(\sum_{i \in I} a_i\right) = \sum_{i \in I} b_i
\]

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Our assumptions on \( \{a_n\} \) and \( \{b_n\} \) imply that \( f \) is defined on \([0, \sum a_i]\) (with values in \([0, \infty]\)) and \( \hat{f} \) is defined on \([0, \infty]\) (with values in \([0, \sum a_n]\)). First let us prove that
\[
f(\hat{f}(x)) = x \quad \text{for every } x \in [0, \infty).
\]
Indeed, let \( x \in (0, \infty) \). Then by Proposition 1, the assumptions on \( \{a_n\} \) imply that \( \hat{f}(x) = \sum_{i \in M} a_i \) where \( M \) is a subset of \( N \). Also if \( x = \sum_{i \in J} b_i \) then \( \hat{f}(x) = \sum_{i \in M} a_i \geq \sum_{i \in J} a_i \). So we can assume that \( M \) is infinite. Set \( \delta_n = \sum_{i \in M_n} a_i \) where \( M_n = M \cap (n, \infty) \) for all \( n \in N \). Then by definition of \( \hat{f} \) one can find \( \sum a_i - \delta_n \leq \sum \{i \in M \setminus \{i \leq n\} \} a_i \leq \sum_{i > n} a_i \). Then by Proposition 1 we obtain
\[
\sum a_i - \sum \{i \in M \setminus \{i \leq n\} \} a_i = \sum_{i \in J_n} a_i,
\]
for some \( J_n \subset N \setminus \{0, 1, \ldots, n\} \). Since \( \{b_n\} \ll \{a_n\} \), we have
\[
\sum b_i = \sum \{i \in M \setminus \{i \leq n\} \} b_i + \sum b_i = x.
\]
Then \( \sum \{i \in M \setminus \{i \leq n\} \} b_i \leq x \) for all \( n \in N \). So \( \sum_{i \in M} b_i \leq x \) holds.

Assume that \( \sum_{i \in M} b_i < x \). Then the third conclusion of Proposition 1 implies that there exists \( K \), a nonempty subset of \( N - M \), such that \( \sum_{i \in K} b_i = x \). So by the definition of \( \hat{f} \) we have
\[
\hat{f}(x) \geq \sum_{i \in M} a_i + \sum_{i \in K} a_i,
\]
which contradicts the fact that \( \hat{f}(x) = \sum_{i \in M} a_i \). Therefore,
\[
f(\hat{f}(x)) = x \quad \text{for all } x \in [0, \infty).
\]

Next we prove that \( \hat{f} \) is strictly increasing. Indeed, let \( x_1, x_2 \in [0, \infty) \) with \( x_1 < x_2 \). Proposition 1(1) implies that \( f(\hat{f}(x_1)) \neq f(\hat{f}(x_2)) \). Put \( \hat{f}(x_1) = \sum_{i \in M_1} a_i \) and \( \hat{f}(x_2) = \sum_{i \in M_2} a_i \). Then by using Proposition 1, one can find \( K \), a nonempty subset of \( N - M_2 \), such that
\[
\sum_{i \in M_1} b_i + \sum_{i \in K} b_i = \sum_{i \in M_2} b_i,
\]
which implies that \( \hat{f}(x_1) = \sum_{i \in M_1} a_i < \hat{f}(x_2) \).
The last step consists of proving that \( \tilde{f} \) is continuous. In order to show that \( \tilde{f} \) is left-continuous (resp. right-continuous) at \( x \in (0, \infty) \), it is enough to prove that for some \( x_n < x \) (resp. \( x_n > x \)) with \( \lim_{n \to \infty} x_n = x \) then \( \lim_{n \to \infty} \tilde{f}(x_n) = \tilde{f}(x) \) because \( \tilde{f} \) is increasing.

Left-continuity is easy to show. Indeed, let \( x \in (0, \infty) \). Then \( \tilde{f}(x) = \sum_{M} a_i \) where \( M \) is an infinite subset of \( N \). Since \( f(\tilde{f}(x)) = x \), we have \( x = \sum_{M} b_i \). Set \( \delta_n = \sum_{i \in M_n} b_i \) and \( x_n = x - \delta_n \) where \( M_n = \{ i \in M ; i > n \} \) for all \( n \in N \). Then clearly we have \( \lim_{n \to \infty} x_n = x \) and \( \lim \tilde{f}(x_n) = \tilde{f}(x) \), since
\[
\sum_{\{i \in M ; i \leq n\}} a_i \leq \tilde{f}(x_n) < \tilde{f}(x).
\]

So \( \tilde{f} \) is left-continuous on \( (0, \infty) \).

To complete the proof of continuity of \( \tilde{f} \), let us show that \( \tilde{f} \) is right-continuous.

Let \( x \in (0, \infty) \), and again set \( \tilde{f}(x) = \sum_{M} a_i \). Since \( x = \sum_{M} b_i \), one can assume that \( N - M \) is infinite. Put \( N - M = \{ k_1, k_2, \ldots \} \) and \( x_n = x + b_{k_n} \) for all \( n \in N \). Then \( \lim_{n \to \infty} x_n = x \). Since \( \{x_n\} \) is decreasing, \( \{\tilde{f}(x_n)\} \) is a decreasing sequence with \( \tilde{f}(x_n) \geq \tilde{f}(x) \) for all \( n \). Then \( \lim_{n \to \infty} \tilde{f}(x_n) = \omega \) exists and \( \omega \geq \tilde{f}(x) \). Our assumptions on \( \{a_n\} \) imply that \( \omega = \sum_{I} a_i \) for some subset \( I \subset N \). Since \( \omega > 0 \), one can assume that \( N - I \) is infinite. Set
\[
\omega_0 = \omega + \sum_{\{i \in N - I ; i > n_0\}} a_i
\]
for \( n_0 \in N \). Then one can find \( l_0 \in N \) such that \( \tilde{f}(x_n) \leq \omega_{n_0} \) for all \( n \geq l_0 \). Since \( \omega \leq \tilde{f}(x_n) \) for all \( n \), we get that
\[
\tilde{f}(x_n) = \sum_{\{i \in I ; i \leq n_0\}} a_i + \sum_{i \in J_n} a_i
\]
for some \( J_n \subset N - \{0, 1, 2, \ldots, n_0\} \) for all \( n \geq l_0 \). Then
\[
x_n = f(\tilde{f}(x_n)) = \sum_{\{i \in I ; i \leq n\}} b_i + \sum_{i \in J_n} b_i
\]
for all \( n \leq l_0 \). In particular, \( \sum_{\{i \in I ; i \leq n_0\}} b_i \leq x_n \) holds for all \( n \geq l_0 \), which implies
\[
x = \lim x_n \geq \sum_{\{i \in I ; i \leq n_0\}} b_i.
\]
Since this is true for all \( n_0 \in N \), we get \( \sum_{I} b_i \leq x \). And because
\[
\omega = \sum_{i \in I} a_i \leq \tilde{f} \left( \sum_{i \in I} b_i \right) \leq \tilde{f}(x) \leq \sum_{i \in I} a_i = \omega,
\]
we get \( \tilde{f}(x) = \omega = \lim \tilde{f}(x_n) \). So the proof of the continuity of \( \tilde{f} \) is complete.

Therefore \( \tilde{f}(0, \infty) \) is an interval. It is not hard to see that \( \tilde{f}(0, \infty) = (0, \sum a_i) \). So there exists \( x \in (0, \infty) \) such that \( \tilde{f}(x) = \sum_{i > n} a_i \) for \( n > 1 \), which implies that \( x = \sum_{i > n} b_i \). This yields a contradiction with \( \sum_{i > n} b_i = \infty \) for all \( n \in N \). So the proof of Proposition 2 is complete.

The next theorem states the main result of this work.
**Theorem 2.** Let \( \{a_n\} \) be a sequence of real numbers such that \( 0 < a_{n+1} \leq a_n \leq \sum_{i>n} a_i \) for all \( n \), and let \( \lim_{n \to \infty} a_n = 0 \). Let \( \{b_n\} \) be a sequence of nonnegative real numbers such that \( \{b_n\} \ll \{a_n\} \). Then there exists \( \alpha \in \mathbb{R} \) such that \( b_n = \alpha a_n \) for all \( n \in \mathbb{N} \).

**Proof.** Consider the function \( f \) defined on \([0, \sum_{i=1}^{\infty} a_i]\) by \( f(\sum_{i=1}^{\infty} a_i) = \sum_{i} b_i \).

If \( \sum a_n \) is divergent then \( f \) is clearly increasing (by Proposition 1). And if \( \sum a_n \) is convergent then \( \sum b_n \) is convergent and again \( f \) is increasing (see [6]). Therefore, \( f \) is almost everywhere differentiable (see [7, p. 96]). Let \( f \) be differentiable at \( x \in (0, \infty) \). Set \( x = \sum_{i} a_i \) with \( I \subset \mathbb{N} \). Define

\[
h_n = \begin{cases} 
   a_n & \text{if } n \in \mathbb{N} - I, \\
   -a_n & \text{if } n \in I.
\end{cases}
\]

Then

\[
\frac{f(x + h_n) - f(x)}{h_n} = \frac{b_n}{a_n}
\]

for all \( n \in \mathbb{N} \). Since \( f \) is differentiable at \( x \), we deduce that \( \lim_{n \to \infty} (b_n/a_n) = \alpha \) exists. Assume that there exists \( n_0 \) such that \( b_{n_0}/a_{n_0} \neq \alpha \). Put

\[
A = \left\{ n \in \mathbb{N} : \frac{b_n}{a_n} \geq \frac{b_{n_0}}{a_{n_0}} \right\}
\]

if \( \frac{b_{n_0}}{a_{n_0}} > \alpha \).

Then \( A \) is a finite set. Therefore, using Proposition 1, there exists an infinite subset \( I \) of \( \mathbb{N} \) such that \( \sum_{i \in A} a_i = \sum_{i \in I} a_i \). So

\[
\sum_{i \in A-I} a_i = \sum_{i \in I-A} a_i,
\]

and since \( \{b_n\} \ll \{a_n\} \), we have

\[
\sum_{i \in A-I} b_i = \sum_{i \in I-A} b_i.
\]

This yields, by the definition of \( A \),

\[
\frac{b_{n_0}}{a_{n_0}} \sum_{i \in A-I} a_i \leq \sum_{i \in A-I} b_i = \sum_{i \in I-A} b_i < \frac{b_{n_0}}{a_{n_0}} \sum_{i \in I-A} a_i.
\]

Therefore, \( \sum_{i \in A-I} a_i = \sum_{i \in I-A} a_i = 0 \), which implies that \( A = I \), contradicting the fact that \( A \) is finite and \( I \) is infinite. So \( b_n = \alpha a_n \) for all \( n \).

We complete the proof by noticing that if \( b_{n_0}/a_{n_0} < \alpha \) then one can set

\[
A = \left\{ n \in \mathbb{N} : \frac{b_n}{a_n} \leq \frac{b_{n_0}}{a_{n_0}} \right\}.
\]

Theorem 2 can be interpreted as a result on purely atomic measures. For the nonatomic case, one can consult [1, 8]. In the next theorem an extension to arbitrary \( \sigma \)-finite measures is discussed. Notice that the finite case is proved in [6].

**Theorem 3.** Let \( \mu \) be a \( \sigma \)-finite measure. Assume that the range of \( \mu \) is an interval. If \( \mu \) is a purely atomic measure, we will assume that the \( \mu \)-measure of the atoms decreases to 0. Then any measure \( \mu' \) such that

\[
\mu(A) = \mu(B) \text{ implies } \mu'(A) = \mu'(B)
\]

is proportional to \( \mu \), i.e., there exists \( \alpha \in \mathbb{R} \) such that \( \mu' = \alpha \mu \).
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