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A QUANTITATIVE DIRICHLET-JORDAN TEST FOR WALSH-FOURIER SERIES

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ABSTRACT. We consider the Walsh-Fourier series $\sum a_k w_k(x)$ of a function f assumed to be of bounded fluctuation on the interval $[0, 1)$. Every function of bounded variation is also of bounded fluctuation on the same interval, but not conversely. We present an estimate for the difference of $f(x)$ at a point $x \in [0, 1)$ and the partial sum of its Walsh-Fourier series in terms of the bounded fluctuation operator. This gives rise to a local convergence result. As special cases, we obtain a Walsh analogue of the Dirichlet-Jordan test and a global convergence result due to Onneweer.

1. INTRODUCTION

We consider the Rademacher orthonormal system $\{r_k(x)\colon k \geq 0\}$ and the Walsh orthonormal system $\{w_k(x)\colon k \geq 0\}$ defined on the unit interval $[0, 1)$, the latter in the Paley enumeration (see [4; 5, p. 1]).

Given a function $f \in L^1[0, 1)$, its Walsh-Fourier series is defined by

$$(1.1) \qquad \sum_{k=0}^{\infty} a_k w_k(x), \qquad a_k := \int_0^1 f(t) w_k t(t)\, dt.$$

The nth partial sum of series (1.1) is

$$s_n(f, x) := \sum_{k=0}^{n-1} a_k w_k(x), \qquad n \geq 1.$$

As is well known,

$$(1.2) \qquad s_n(f, x) = \int_0^1 f(x \dotplus t) D_n(t)\, dt,$$

where $\dotplus$ means dyadic addition and $D_n(t) := \sum_{k=0}^{n-1} w_k(t)$, $n \geq 1$, is the Walsh-Dirichlet kernel of order n.

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For these and further definitions, notations, and properties of the Walsh system, we refer the reader to [5].

2. Main results

By a dyadic interval in $[0, 1)$ we mean an interval of the form

$$I(j, k) := [j2^{-k}, (j+1)2^{-k}), \qquad 0 \le j < 2^k \text{ and } k \ge 0.$$

For a function f defined on $I(j, k)$, we set

$$\omega(f, I(j, k)) := \sup\{|f(x \dotplus t) - f(x)| : x \in I(j, k) \text{ and } 0 \le t < 2^{-k}\}.$$

Now, f is said to be of bounded fluctuation on a dyadic interval $I := I(j_0, k_0)$, where $0 \le j_0 < 2^{k_0}$ and $k_0 \ge 0$, if

$$\mathscr{FL}(f, I) := \sup_{k \ge k_0} \sum_{j=j_0 2^{k-k_0}}^{(j_0+1)2^{k-k_0}-1} |\omega(f, I(j, k))| < \infty.$$

The quantity $\mathscr{FL}(f, I)$ is called the total fluctuation of f on I. Clearly, every function of bounded variation on I is also of bounded fluctuation on the same I, but not conversely.

The notion of bounded fluctuation on the whole interval $[0, 1)$ is due to Onneweer and Waterman [3].

By (1.2), we may write

$$s_n(f, x) - f(x) = \int_0^1 g_x(t) D_n(t)\, dt, \tag{2.1}$$

where here and in the sequel we use the notation

$$g_x(t) := f(x \dotplus t) - f(x), \qquad x, t \in [0, 1).$$

Our main result reads as follows.

Theorem 1. *If f is of bounded fluctuation on $[0, 1)$, then for any $n = 2^k + m$ with $0 \le m < 2^k$ and $k \ge 0$, and for any $x \in [0, 1)$ we have*

$$|s_n(f, x) - f(x)| \le 2^{-k} \sum_{j=0}^{k} 2^j \mathscr{FL}(g_x, I(0, j)). \tag{2.2}$$

It is plain that if

$$\lim_{t \to +0} g_x(t) = 0 \tag{2.3}$$

for some $x \in [0, 1)$, and if f is of bounded fluctuation on a dyadic interval containing x, then

$$\lim_{j \to \infty} \mathscr{FL}(g_x, I(0, j)) = 0. \tag{2.4}$$

Thus, Theorem 1 implies

Corollary 1. *If f is of bounded fluctuation on $[0, 1)$, and if condition (2.3) is satisfied for some $x \in [0, 1)$, then*

$$\lim_{n\to\infty} s_n(f, x) = f(x). \tag{2.5}$$

Relation (2.5) was proved by Walsh [6] in the case when f is of bounded variation. Its trigonometric analogue is known as the Dirichlet-Jordan test (see, e.g., [7, Vol. 1, p. 57]). The first quantitative version of the Dirichlet-Jordan test was proved by Bojanić [1].

We note that if f is uniformly W-continuous on $[0, 1)$ (concerning this notion see [5, pp. 9–11]), then relation (2.3) and a fortiori (2.4) hold uniformly in x. In this way, Theorem 1 yields

Corollary 2. *If f is uniformly W-continuous and of bounded fluctuation on $[0, 1)$, then we have (2.5) uniformly on $[0, 1)$.*

This result was first proved by Onneweer [2] in the case when f is of bounded variation.

Actually, we will prove Theorem 1 in a sharper form as follows.

Theorem 2. *If f is of bounded fluctuation on $[0, 1)$, then for any*

$$n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_p}, \qquad k_1 > k_2 > \cdots > k_p \geq 0, \tag{2.6}$$

and for any $x \in [0, 1)$ we have

$$|s_n(f, x) - f(x)| \leq \mathscr{FL}(g_x, I(0, k_1)) + \sum_{j=2}^{p} 2^{k_j-k_1-1}\mathscr{FL}(g_x, I(0, k_j)). \tag{2.7}$$

In case $p = 1$, the empty sum equals 0 as usual.

We make one more remark in the particular case when f is of bounded variation on $[0, 1]$ with the agreement that $f(1) := f(0)$. More generally, we agree to set $f(x+1) := f(x)$. Denote by $V_a^b(f)$ the total variation of f on the interval $[a, b]$. (This time the end points a and b are not necessarily dyadic rational numbers.) Since

$$\mathscr{FL}(g_x, I(0, j)) \leq V_0^{2^{-j}}(g_x) \leq V_x^{x+2^{-j}}(f),$$

relation (2.2) becomes

$$|s_n(f, x) - f(x)| \leq 2^{-k} \sum_{j=0}^{k} 2^j V_x^{x+2^{-j}}(f),$$

while (2.7) becomes

$$|s_n(f, x) - f(x)| \leq V_x^{x+2^{-k_1}}(f) + \sum_{j=2}^{p} 2^{k_j-k_1-1} V_x^{x+2^{-k_j}}(f).$$

3. Proof of Theorem 2

We start with the well-known identity (see, e.g., [5, p. 46])

$$D_n(t) = D_{2^{k_1}}(t) + r_{k_1}(t) D_{m_1}(t), \tag{3.1}$$

where n is given by (2.6) and

$$m_1 := n - 2^{k_1} = 2^{k_2} + 2^{k_3} + \cdots + 2^{k_p}.$$

Thus by (2.1),

$$\begin{aligned} s_n(f, x) - f(x) &= \int_0^1 g_x(t) D_{2^{k-1}}(t)\, dt + \int_0^1 g_x(t) r_{k_1}(t) D_{m_1}(t)\, dt \\ &=: A_1 + B_1, \qquad \text{say}. \end{aligned} \tag{3.2}$$

Since

$$D_{2^{k_1}}(t) = \begin{cases} 2^{k_1} & \text{if } t \in [0, 2^{-k_1}), \\ 0 & \text{if } t \in [2^{-k_1}, 1) \end{cases}$$

(see, e.g., [5, p. 7]) and since $g_x(0) = 0$, we find that

$$\begin{aligned} |A_1| &= 2^{k_1} \left| \int_0^{2^{-k_1}} g_x(t)\, dt \right| \le 2^{k_1} \int_0^{2^{-k_1}} |g_x(t) - g_x(0)|\, dt \\ &\le 2^{k_1} \int_0^{2^{-k_1}} \mathscr{H}(g_x, I(0, k_1))\, dt = \mathscr{H}(g_x, I(0, k_1)). \end{aligned} \tag{3.3}$$

Next, we will estimate B_1. To this effect, we keep in mind the following elementary facts:

(i) $D_{m_1}(t)$ takes on a constant value on each dyadic interval $I(j_1, k_1)$, where $0 \le j_1, m_1 < 2^{k_1}$;

(ii) $I(j_1, k_1) = I(2j_1, k_1 + 1) \cup I(2j_1 + 1, k_1 + 1)$;

(iii)

$$r_{k_1}(t) = \begin{cases} 1 & \text{if } t \in I(2j_1, k_1 + 1), \\ -1 & \text{if } t \in I(2j_1 + 1, k_1 + 1); \end{cases}$$

(iv) $u := t \dotplus 2^{-k_1-1}$ is a one-to-one mapping of $I(2j_1, k_1 + 1)$ onto $I(2j_1 + 1, k_1 + 1)$; in each case we assume that $0 \le j_1 < 2^{k_1}$ and $k_1 \ge 0$.

Thus, by (i)–(iv),

$$\begin{aligned} B_1 &= \sum_{j_1=0}^{2^{k_1}-1} D_{m_1}(j_1 2^{-k_1}) \left\{ \int_{I(2j_1, k_1+1)} g_x(t)\, dt - \int_{I(2j_1+1, k_1+1)} g_x(t)\, dt \right\} \\ &= \sum_{j_1=0}^{2^{k_1}-1} D_{m_1}(j_1 2^{-k_1}) \int_{I(2j_1, k_1+1)} \{g_x(t) - g_x(t \dotplus 2^{-k_1-1})\}\, dt \\ &= \sum_{j_1=0}^{2^{k_1}-1} \int_{I(2j_1, k_1+1)} \{g_x(t) - g_x(t \dotplus 2^{-k_1-1})\} D_{m_1}(t)\, dt. \end{aligned} \tag{3.4}$$

We note that the integration domain occupies only one half of the interval $[0, 1)$. We introduce a second difference function as follows

$$h_x(t) := g_x(t) - g_x(t \dotplus 2^{-k_1-1}). \tag{3.5}$$

Similarly to (3.1), we may write that

$$D_{m_1}(t) = D_{2^{k_2}}(t) + r_{k_2}(t) D_{m_2}(t),$$

where

$$m_2 := m_1 - 2^{k_2} = 2^{k_3} + \cdots + 2^{k_p}$$

(cf. (2.6)). From (3.4) and (3.5) it follows that

$$B_1 = \sum_{j_1=0}^{2^{k_1}-1} \int_{I(2j_1, k_1+1)} h_x(t) D_{2^{k_2}}(t)\, dt + \sum_{j_1=0}^{2^{k_1}-1} \int_{I(2j_1, k_1+1)} h_x(t) r_{k_2}(t) D_{m_2}(t)\, dt =: A_2 + B_2, \quad \text{say}. \tag{3.6}$$

Using an argument analogous to the one occurring in (3.3) and the fact that $u := t \dotplus 2j_1 2^{-k_1-1}$ is a one-to-one mapping of $I(0, k_1+1)$ onto $I(2j_1, k_1+1)$, yields

$$\begin{aligned} |A_2| &= \left| 2^{k_2} \sum_{j_1=0}^{2^{k_1-k_2-1}} \int_{I(2j_1, k_1+1)} h_x(t)\, dt \right| \\ &= \left| 2^{k_2} \int_{I(0, k_1+1)} \left\{ \sum_{j_1=0}^{2^{k_1-k_2}-1} h_x(t \dotplus 2j_1 2^{-k_1-1}) \right\} dt \right| \\ &\le 2^{k_2} \int_{I(0, k_1+1)} \left\{ \sum_{j_1=0}^{2^{k_1-k_2}-1} |g_x(t \dotplus 2j_1 2^{-k_1-1}) - g_x(t \dotplus (2j_1+1) 2^{-k_1-1})| \right\} dt \\ &\le 2^{k_2} \int_{I(0, k_1+1)} \mathscr{H}(g_x, I(0, k_2))\, dt = 2^{k_2-k_1-1} \mathscr{H}(g_x, I(0, k_2)). \end{aligned} \tag{3.7}$$

Now, we turn to B_2. Relying upon properties (i)–(iv), we can deduce that (cf. (3.4))

$$\begin{aligned} B_2 &= \sum_{j_1=0}^{2^{k_1}-1} \sum_{j_2=0}^{2^{k_2}-1} D_{m_2}(j_2 2^{-k_2}) \left\{ \int_{I(2j_1, k_1+1) \cap I(2j_2, k_2+1)} h_x(t)\, dt \right. \\ &\qquad \left. - \int_{I(2j_1, k_1+1) \cap I(2j_2+1, k_2+1)} h_x(t)\, dt \right\} \\ &= \sum_{j_1=0}^{2^{k_1}-1} \sum_{j_2=0}^{2^{k_2}-1} \int_{I(2j_1, k_1+1) \cap I(2j_2, k_2+1)} \{h_x(t) - h_x(t \dotplus 2^{-k_2-1})\} D_{m_2}(t)\, dt \\ &= \sum_{j_2=0}^{2^{k_2}-1} \sum_{j_1=2j_2 2^{k_1-k_2-1}}^{(2j_2+1)2^{k_1-k_2-1}-1} \int_{I(2j_1, k_1+1)} \{h_x(t) - h_x(t \dotplus 2^{-k_2-1})\} D_{m_2}(t)\, dt. \end{aligned} \tag{3.8}$$

We note that this time the integration domain occupies only one fourth of the interval $[0, 1)$. We introduce a third difference function

$$\eta_x(t) := h_x(t) - h_x(t \dotplus 2^{-k_2-1}). \tag{3.9}$$

Repeating the above reasoning, from (3.8) and (3.9) it follows that

$$
\begin{aligned}
B_2 &= \sum_{j_1=0}^{2^{k_1}-1} \sum_{j_2=0}^{2^{k_2}-1} \int_{I(2j_1,\,k_1+1)\cap I(2j_2,\,k_2+1)} \eta_x(t) D_{2^{k_3}}(t)\,dt \\
&\quad + \sum_{j_1=0}^{2^{k_1}-1} \sum_{j_2=0}^{2^{k_2}-1} \int_{I(2j_1,\,k_1+1)\cap I(2j_2,\,k_2+1)} \eta_x(t) r_{k_3}(t) D_{m_3}(t)\,dt \\
&=: A_3 + B_3, \qquad \text{say},
\end{aligned} \tag{3.10}
$$

where $m_3 := m_2 - 2^{k_3} = 2^{k_4} + \cdots + 2^{k_p}$.

Analogously to the last equality in (3.8), hence we may conclude that

$$
\begin{aligned}
|A_3| &= 2^{k_3} \left| \sum_{j_1=0}^{2^{k_1-k_3}-1} \sum_{j_2=0}^{2^{k_2-k_3}-1} \int_{I(2j_1,\,k_1+1)\cap I(2j_2,\,k_2+1)} \eta_x(t)\,dt \right| \\
&= 2^{k_3} \left| \sum_{j_2=0}^{2^{k_2-k_3}-1} \sum_{j_1=2j_2 2^{k_1-k_2-1}}^{(2j_2+1)2^{k_1-k_2-1}-1} \int_{I(2j_1,\,k_1+1)} \eta_x(t)\,dt \right| \\
&\le 2^{k_3} \int_{I(0,\,k_1+1)} \left\{ \sum_{j_2=0}^{2^{k_2-k_3-1}-1} \sum_{j_1=2j_2 2^{k_1-k_2-1}}^{(2j_2+1)2^{k_1-k_2-1}-1} |\eta_x(t \dotplus 2j_1 2^{-k_1-1})| \right\} dt.
\end{aligned} \tag{3.11}
$$

By (3.5) and (3.9),

$$
\begin{aligned}
|\eta_x(t \dotplus 2j_1 2^{-k_1-1})| &\le |g_x(t \dotplus 2j_1 2^{-k_1-1}) - g_x(t \dotplus (2j_1+1)2^{-k_1-1})| \\
&\quad + |g_x(t \dotplus 2^{-k_2-1} \dotplus 2j_1 2^{-k_1-1}) - g_x(t \dotplus 2^{-k_2-1} \dotplus (2j_1+1)2^{-k_1-1})|.
\end{aligned}
$$

Substituting this for the integrand in (3.11) results that

$$
\begin{aligned}
|A_3| &\le 2^{k_3} \int_{I(0,\,k_1+1)} \mathscr{H}(g_x,\, I(0,\, k_3))\,dt \\
&= 2^{k_3-k_1-1} \mathscr{H}(g_x,\, I(0,\, k_3)).
\end{aligned} \tag{3.12}
$$

Furthermore, it is also not difficult to see that

$$
\begin{aligned}
B_3 &= \sum_{j_1=0}^{2^{k_1}-1} \sum_{j_2=0}^{2^{k_2}-1} \sum_{j_3=0}^{2^{k_3}-1} \\
&\qquad \times \int_{I(2j_1,\,k_1+1)\cap I(2j_2,\,k_2+1)\cap I(2j_3,\,k_3+1)} \{\eta_x(t) - \eta_x(t \dotplus 2^{-k_3-1})\} D_{m_3}(t)\,dt \\
&= \sum_{j_3=0}^{2^{k_3-1}} \sum_{j_2=2j_3 2^{k_2-k_3-1}}^{(2j_3+1)2^{k_2-k_3-1}-1} \sum_{j_1=2j_2 2^{k_1-k_2-1}}^{(2j_2+1)2^{k_1-k_2-1}-1} \\
&\qquad \times \int_{I(2j_1,\,k_1+1)} \{\eta_x(t) - \eta_x(t \dotplus 2^{-k_3-1})\} D_{m_3}(t)\,dt,
\end{aligned}
$$

where the integration domain now occupies only one eighth of the interval $[0, 1)$.

By an induction argument we can proceed up to $B_{p-1} = A_p$ (observe that $B_p = 0$) in the same manner as above. Owing to the difficulties in notation, we omit the details.

Finally, we combine (3.2), (3.3), (3.6), (3.7), (3.10), (3.12) (and the analogous estimates of $|A_q|$ for $q = 4, 5, \dots, p$) and obtain (2.7). Thus, the proof of Theorem 2 is complete.

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