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TWO WEIGHT Φ-INEQUALITIES FOR THE HARDY OPERATOR, HARDY-LITTLEWOOD MAXIMAL OPERATOR, AND FRACTIONAL INTEGRALS

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Abstract. Suppose Φ is an appropriate Young's function and $w(x)$, $v(x)$ are nonnegative locally integrable functions. Let T denote one of three linear operators of special importance that map suitable functions on R^n into functions on R^n.

For the Hardy operator T, we study the inequality

$$\int_0^\infty \Phi(|Tf(x)|)w(x)\,dx \le C\int_0^\infty \Phi(|f(x)|)v(x)\,dx$$

and for the Hardy-Littlewood maximal operator or fractional integrals T, we discuss the inequalities

$$\int_{R^n} \Phi(|T(fv)(x)|)w(x)\,dx \le C\int_{R^n} \Phi(|f(x)|)v(x)\,dx.$$

In all cases we obtain the necessary and sufficient conditions.

1. Introduction

We shall be concerned with integral inequalities of the form

$$\int_{R^n} \Phi|Tf(x)|)\,dw \le C\int_{R^n} \Phi(|f(x)|)\,d\mu, \tag{1.1}$$

where dw, $d\mu$ are positive Borel measures on R^n, Φ is an even Young's function on R with $\Phi(0)=0$, and T is one of three linear operators of special importance that map suitable functions on R^n into functions on R^n. Apart from their intrinsic interest, such inequalities are important in application, since they imply the boundedness of T as a map between the associated Orlicz spaces. It is of particular interest to obtain estimates for the best constant C in (1.1).

The first of the cases we consider is the Hardy operator

$$Tf(x) = \int_0^x f(t)\,dt, \qquad x \in R^+ = (0,\infty). \tag{1.2}$$

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Let ϕ be a nondecreasing right continuous function on R^+ with $\phi(0+) = 0$, and define Φ to be Young's function $\Phi(t) = \int_0^{|t|} \phi(s)\,ds$. Let ϕ^{-1} be the right continuous inverse of ϕ and set $\Psi(t) = \int_0^{|t|} \phi^{-1}(s)\,ds$, the complementary function of Φ. (See [3] or [4] for the details of Young's function and its complement.) Our first result is

Theorem 1. *Let w, v be nonnegative, locally integrable functions on R^+, and suppose that Φ is a Young's function such that both Φ and its Young's complement Ψ satisfy the Δ_2-condition, i.e.,*

$$\Phi(2t) \le A\Phi(t), \quad \Psi(2t) \le B\Psi(t), \qquad \text{for all } t \in R^+. \tag{1.3}$$

Then, with T defined in (1.2), *there exists a constant C, independent of f, such that*

$$\int_0^\infty \Phi(Tf(x))w(x)\,dx \le C\int_0^\infty \Phi(f(x))v(x)\,dx \tag{1.4}$$

if and only if there exists a constant K such that

$$\left(\int_x^\infty \varepsilon w(t)\,dt\right)\phi\left(\int_0^x \phi^{-1}\left(\frac{1}{\varepsilon v(t)}\right)dt\right) \le K \tag{1.5}$$

for all $\varepsilon > 0$ and all $x > 0$.

Furthermore, if we denote the best constants in (1.4) *and* (1.5) *by C and K, respectively, then there are positive constants C_1, C_2, $0 < \lambda_1 \le 1 \le \lambda_2 \le \lambda_3 < \infty$, and $0 < \theta \le 1$ depending only on Φ such that*

$$C_1K \le C \le \begin{cases} C_2K^{\lambda_1} & \text{when } K \ge 1, \\ C_2K^{\lambda_2} & \text{when } \theta \le K < 1, \\ C_2K^{\lambda_3} & \text{when } 0 < K < \theta. \end{cases} \tag{1.6}$$

The special case $\Phi(t) = |t|^p$ $(1 < p < \infty)$ of Theorem 1 is the well-known result of Muckenhoupt [5], which generalized the classical Hardy inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p\,dx \quad \text{for } f(x) \ge 0 \text{ and } p > 1.$$

Muckenhoupt's theorem is that given $p > 1$ and a couple of nonnegative locally integrable weight functions $w(x)$ and $v(x)$, the inequality

$$\left(\int_0^\infty \left|\int_0^x f(t)dt\right|^p w(x)\,dx\right)^{1/p} \le C\left(\int_0^\infty |f(x)|^p v(x)\,dx\right)^{1/p}$$

holds if and only if

$$\sup_{r>0}\left(\int_r^\infty w(x)\,dx\right)^{1/p}\left(\int_0^r v(x)^{-1/(p-1)}\,dx\right)^{1/p'} = K < \infty$$

and $K \le C \le p^{1/p'}(p')^{2/p'}K$, where $1/p + 1/p' = 1$. In these cases the constants K appearing in (1.6) and in Muckenhoupt's theorem are equivalent.

Our second theorem concerns fractional integrals in R^n. Following Sawyer in [8], we work with general convolution operators of the form

$$T(f\mu)(x) = K^*(f\mu)(x) = \int_{R^n} K(x-y)f(y)\,d\mu(y), \tag{1.7}$$

where the convolution kernel $K(x)$ is a positive lower semicontinuous radial function decreasing in $|x|$ and satisfying the growth condition $K(x) \le CK(2x)$ for every $x \in R^n$. We shall write χ_E for the characteristic function of a set $E \subset R^n$, and put $|E|_\mu = \int_E d\mu$. The Luxemberg norm $\|\cdot\|_{\Phi(\mu)}$ on the Orlicz space $L_{\Phi(\mu)} = \{f\colon \int_{R^n} \Phi(|f(x)|)\, d\mu(x) < \infty\}$ is given by

$$\|f\|_{\Phi(\mu)} = \inf\left\{\theta > 0\colon \int_{R^n} \Phi\left(\frac{|f(x)|}{\theta}\right) d\mu(x) \le 1\right\}. \tag{1.8}$$

With these notations we have

Theorem 2. *Let Φ be a Young's function as in Theorem 1, T be as defined in (1.7), and dw, $d\mu$ be positive Borel measures on R^n. Then in order that there exists a constant C independent of f such that*

$$\int_{R^n} \Phi(T(f\mu)(x))\, dw(x) \le C \int_{R^n} \Phi(f(x))\, d\mu(x) \tag{1.9}$$

for all $f \ge 0$, it is necessary and sufficient that there is a constant C such that both

$$\int_{R^n} \Phi(T(\varepsilon\chi_Q\mu)(x))\, dw(x) \le C\Phi(\varepsilon)|Q|_\mu < \infty \tag{1.10}$$

and

$$\|T(\chi_Q w)\|_{\Psi(\varepsilon\mu)} \le C\|\chi_Q\|_{\Psi(\varepsilon w)} < \infty \tag{1.11}$$

hold for all $\varepsilon > 0$ and all dyadic cubes Q.

In [7, 8] Sawyer and Wheeden discussed fractional integrals on weighted L^p spaces. Our Theorem 2 is a generalization of Sawyer's result in [8].

Finally, we discuss the celebrated Hardy-Littlewood maximal function

$$Mf(x) = \sup_{x\in Q,\ \text{cube}} \frac{1}{|Q|} \int_Q |f(y)|\, dy. \tag{1.12}$$

In fact, we shall deal with the following variant of the maximal function due to Fefferman and Stein (see [1]):

$$Tf(x, t) = Mf(x, t) = \sup_Q \frac{1}{|Q|} \int_Q |f(y)|\, dy, \qquad x \in R^n,\ t \ge 0, \tag{1.13}$$

where the supremum is taken over all the cubes Q in R^n, containing x and having side length $\imath(Q) \ge t$. For this operator we obtain

Theorem 3. *Suppose $d\mu$ is a positive Borel measure on $R^{n+1}_+ = \{(x, t)\colon x \in R^n,\ t \ge 0\}$ and $w(x)$ is a nonnegative locally integrable weight function on R^n. Let Φ be a Young's function that satisfies the Δ'-condition (see [4])*

$$\Phi(uv) \le C\Phi(u)\Phi(v) \quad \text{for all } u, v > 0 \tag{1.14}$$

and its complement Ψ still obey the Δ_2-condition. Then, for T defined in (1.13), in order that there exists a constant C independent of f such that

$$\int_{R^{n+1}_+} \Phi(T(fw)(x, t))\, d\mu(x, t) \le C \int_{R^n} \Phi(f(x))w(x)\, dx, \tag{1.15}$$

it is necessary and sufficient that there is a constant C *such that*

$$\int_{\widehat{Q}} \Phi(T(w\chi_Q)(x, t))\, d\mu(x, t) \leq C|Q|_w < \infty \tag{1.16}$$

for all cubes Q, *where* $\widehat{Q} = \{(x, t): x \in Q,\ 0 \leq t \leq \imath(Q)\}$.

It is well known that the two-weight norm inequality for the Hardy maximal operator defined in (1.12) has been characterized by Sawyer [9]. Later on, a similar result for the Fefferman-Stein maximal function was obtained in [6]. Our Theorem 3 extends their results.

In this paper we need the following elementary properties of Young's function Φ and its complement Ψ (see [4] or [3]):

$$t < \Phi^{-1}(t)\Psi^{-1}(t) \leq 2t \tag{1.17}$$

and

$$\Phi(t) \sim t\phi(t), \qquad \Psi(t) \sim t\phi^{-1}(t), \tag{1.18}$$

when Φ and Ψ both satisfy the Δ_2-condition. The symbol '$\sim$' means the ratio of the two sides is bounded between absolute positive constants.

2. Proof of theorems

Proof of Theorem 1. *Necessity.* If $f \geq 0$ and $\operatorname{supp} f \subset [0, x]$ for given x, then from (1.4) it follows that

(2.1)
$$\Phi\left(\int_0^x f(t)\, dt\right) \int_x^\infty w(t)\, dt \leq \int_0^\infty \Phi(Tf(t))w(t)\, dt \leq C \int_0^x \Phi(f(t))v(t)\, dt.$$

Set $f(t) = \phi^{-1}(1/\varepsilon v(t))\chi_{[0,x]}(t)$. If $\int_0^x f(t)\, dt = 0$, we have

$$\left(\int_x^\infty \varepsilon w\right) \phi\left(\int_0^x \phi^{-1}(1/\varepsilon v)\right) = 0.$$

If $\int_0^x f(t)\, dt = \infty$, then $\int_0^x \Psi(1/\varepsilon v(t))\varepsilon v(t)\, dt = \infty$ because of (1.18). This implies that there is a nonnegative $g(t)$ such that $\int \Phi(g)\, \varepsilon v < \infty$ and $\int_0^x g(t)(1/\varepsilon v(t))\varepsilon v(t)\, dt = \infty$. Thus inequality (1.4) ensures that for arbitrary $\lambda > 0$,

$$\begin{aligned} |[x, \infty]|_{\varepsilon w} &\leq |\{s: Tg(s) > \lambda\}|_{\varepsilon w} = |\{s: \Phi(Tg(s)) > \Phi(\lambda)\}|_{\varepsilon w} \\ &\leq \frac{1}{\Phi(\lambda)} \int_0^\infty \Phi(Tg)\varepsilon w \leq \frac{C}{\Phi(\lambda)} \int_0^\infty \Phi(g)\varepsilon v = \frac{C}{\Phi(\lambda)}. \end{aligned}$$

Therefore $|[x, \infty]|_{\varepsilon w} = 0$. Finally, we consider the case $0 < \int_0^x f(t)\, dt < \infty$. Using (1.18) and (2.1), we obtain

$$\begin{aligned} &\left(\int_0^x \phi^{-1}\left(\frac{1}{\varepsilon v}\right)\right) \phi\left(\int_0^x \phi^{-1}\left(\frac{1}{\varepsilon v}\right)\right) \int_x^\infty w \leq C \int_x^\infty \Phi(Tf)w \\ &\quad \leq C \int_0^\infty \Phi(f)v \leq C \int_0^x \phi^{-1}\left(\frac{1}{\varepsilon v}\right) \phi\left(\phi^{-1}\left(\frac{1}{\varepsilon v}\right)\right) v. \end{aligned} \tag{2.2}$$

This is (1.5).

Sufficiency. We need the following inequality: We will show that condition (1.5) implies that

$$\left\| \frac{\chi_{[0,x]}}{\varepsilon v} \right\|_{\Psi(\varepsilon v)} \le C\Phi^{-1}\left(\frac{1}{\varepsilon \int_x^\infty w} \right) \tag{2.3}$$

holds for all $\varepsilon > 0$ and all $x > 0$.

Indeed, from (1.5) it follows that

$$\frac{\int_0^x \phi^{-1}(1/\varepsilon v)}{\phi^{-1}(1/\varepsilon \int_x^\infty w)} \le C. \tag{2.4}$$

Setting $\eta = \eta_{x,\varepsilon}$ such that $\eta\varepsilon\Psi(1/\varepsilon \int_x^\infty w) \int_x^\infty w = 1$, we have

$$\frac{1}{\eta} \sim \phi^{-1}\left(\frac{1}{\varepsilon \int_x^\infty w} \right) \sim \Phi^{-1}\left(\frac{1}{\eta\varepsilon \int_x^\infty w} \right) \tag{2.5}$$

by use of (1.17) and (1.18). Substituting (2.5) into (2.4) and using (1.18), we obtain

$$\int_0^x \Psi\left(\frac{1}{\varepsilon v(t)} \right) \eta\varepsilon v(t)\, dt \le C,$$

thus

$$\left\| \frac{\chi_{[0,x]}}{\varepsilon\eta v} \right\|_{\Psi(\eta\varepsilon v)} \le \frac{C}{\eta} \le C\Phi^{-1}\left(\frac{1}{\eta\varepsilon \int_x^\infty w} \right).$$

Since $\eta\varepsilon = 1/(\Psi(1/\varepsilon \int_x^\infty w) \int_x^\infty w)$ is a continuous function of ε and takes values from 0 to ∞, we can have $\eta\varepsilon = \delta$ for δ given arbitrarily, and we conclude (2.3).

Now we can assume $f \ge 0$ and choose x_k such that $Tf(x_k) = \int_0^{x_k} f(t)\, dt = 2^k$ for all integers k if it is possible. Then we have

$$\begin{aligned} \int_0^\infty \Phi(Tf)w &\le \sum_k \Phi(2^{k+1}) \int_{x_k}^{x_{k+1}} w \le C \sum_k \Phi(2^{k-1}) \int_{x_k}^\infty w \\ &= C \sum_k \Phi\left(\int_{x_{k-1}}^{x_k} f(t)\, dt \right) \int_{x_k}^\infty w. \end{aligned} \tag{2.6}$$

Let k be fixed and set $f_k = f(t)\chi_{(x_{k-1}, x_k]}(t)$. The Hölder inequality in Orlicz spaces shows that for any $\varepsilon_k > 0$ given,

$$\int_{x_{k-1}}^{x_k} f \le C\|f_k\|_{\Phi(\varepsilon_k v)} \left\| \frac{\chi_{[0,x_k]}}{\varepsilon_k v} \right\|_{\Psi(\varepsilon_k v)}.$$

Choose ε_k such that $\int \Phi(f_k)\varepsilon_k v = 1$. Then $\|f_k\|_{\Phi(\varepsilon_k v)} \le 1$. Thus, from (2.3) it follows that

$$\int_{x_{k-1}}^{x_k} f \le C\Phi^{-1}\left(\frac{1}{\varepsilon_k \int_{x_k}^\infty w} \right),$$

therefore,

$$\Phi\left(\int_{x_{k-1}}^{x_k} f \right) \le \frac{C}{\varepsilon_k \int_{x_k}^\infty w} = C\frac{\int \Phi(f_k)v}{\int_{x_k}^\infty w}. \tag{2.7}$$

Substituting (2.7) into (2.6) and observing that the supports of the f_k are disjoint, we have

$$\int_0^\infty \Phi(Tf)w \le C\sum_k \int \Phi(f_k)v = C\int_0^\infty \Phi(f)v.$$

To discuss the relation between the best constant C in (1.4) and K in (1.5), we introduce some indices concerning Φ and Ψ. Define

$$\begin{aligned} q_\Phi &= \inf_{u>0}\frac{u\phi(u)}{\Phi(u)} \le \sup_{u>0}\frac{u\phi(u)}{\Phi(u)} = p_\Phi, \\ p'_\Phi &= \inf_{u>0}\frac{u\phi^{-1}(u)}{\Psi(u)} \le \sup_{u>0}\frac{u\phi^{-1}(u)}{\Psi(u)} = q'_\Phi. \end{aligned} \tag{2.8}$$

Then $1/p_\Phi + 1/p'_\Phi = 1$ and $1/q_\Phi + 1/q'_\Phi = 1$. Since both Φ and Ψ satisfy the Δ_2-condition, we have $1 < q_\Phi, p_\Phi < \infty$ (see [4]). Thus equalities (2.8) imply that

$$\begin{aligned} \lambda^{q_\Phi}\Phi(u) \le \Phi(\lambda u) \le \lambda^{p_\Phi}\Phi(u) \quad &\text{when } \lambda \ge 1, \\ \lambda^{p_\Phi}\Phi(u) \le \Phi(\lambda u) \le \lambda^{q_\Phi}\Phi(u) \quad &\text{when } \lambda < 1 \end{aligned} \tag{2.9}$$

hold for any $u > 0$. Also analogous inequalities hold for Ψ. By use of these inequalities, (1.17), and (1.18), it is easy to obtain the estimate (1.6). For example, $C_1 = q_\Phi/p_\Phi$ follows from the derivation of (2.2) since $q_\Phi \le u\phi(u)/\Phi(u) \le p_\Phi$. Moreover, the three power indices in (1.6) are $\lambda_i = 1$ $(i = 1, 2, 3)$ when $\Phi(u) = |u|^p$. This completes the proof of Theorem 1.

Proof of Theorem 2. If (1.9) holds, it is obvious that the dual norm inequality

$$\|T(gw)\|_{\Psi(\varepsilon\mu)} \le C\|g\|_{\Psi(\varepsilon w)} \tag{2.10}$$

is true for all $\varepsilon > 0$ and all $g \ge 0$. With $f = \varepsilon\chi_Q$ in (1.9) and $g = \chi_Q$ in (2.10), we obtain (1.10) and (1.11).

Conversely, suppose (1.10) and (1.11) hold and, without loss of generality, that f is nonnegative and bounded with compact support. For every integer k we consider the open set $\Omega_k = \{T(f\mu) > 2^k\}$. From the Whitney decomposition lemma (cf. [8]), we obtain a sequence of dyadic cubes $\{Q_j^k\}$ satisfying

(2.11)

(1) $\Omega_k = \bigcup_j Q_j^k$ and $(Q_j^k)^0 \cap (Q_i^k)^0 = \varnothing$ for $i \ne j$;
(2) $RQ_j^k \subset \Omega_k$ and $3RQ_j^k \cap \Omega_k^c \ne \varnothing$ for all k, j;
(3) $\sum_j \chi_{3Q_j^k} \le C\chi_{\Omega_k}$ for all k;
(4) the number of cubes Q_s^k intersecting a fixed cube $3Q_j^k$ is at most C;
(5) $Q_j^k \subsetneqq Q_i^s$ implies $k > s$.

In (2.11), E^0 is the interior of set E, and RQ denotes the cube concentric with Q with R times the side length, the constant R in (2.11)(2) being ≥ 3 and depending only on the dimension n.

As in [8], we choose an integer $m \ge 2$ sufficiently large, depending only on the growth condition of $K(x)$ defined in (1.7). Write $E_j^k = Q_j^k \cap (\Omega_{k+m-1}\backslash\Omega_{k+m})$

for all (k, j). It is established in [8] that

$$\begin{aligned}|E_j^k|_w &\le 2^{-k}\left(\int_{3Q_j^k\setminus\Omega_{k+m}} fT(\chi_{E_j^k}w)\,d\mu + \int_{3Q_j^k\cap\Omega_{k+m}} fT(\chi_{E_j^k}w)\,d\mu\right)\\ &= 2^{-k}(\sigma_j^k + r_j^k).\end{aligned}\tag{2.12}$$

Let $\beta \in (0, 1)$, to be chosen later, and define

$$\begin{aligned}E &= \{(k, j): |E_j^k|_w \le \beta|Q_j^k|_w\},\\ F &= \{(k, j): |E_j^k|_w > \beta|Q_j^k|_w \text{ and } \sigma_j^k > r_j^k\},\\ G &= \{(k, j): |E_j^k|_w > \beta|Q_j^k|_w \text{ and } \sigma_j^k \le r_j^k\}.\end{aligned}$$

Thus we have

$$\begin{aligned}\int \Phi(T(f\mu))\,dw &\le \sum_k \Phi(2^{k+m})|\Omega_{k+m-1}\setminus\Omega_{k+m}|_w \le C\sum_{k,j}|E_j^k|_w\Phi(2^k)\\ &= C\left(\sum_{(k,j)\in E} + \sum_{(k,j)\in F} + \sum_{(k,j)\in G}\right)|E_j^k|_w\Phi(2^k)\\ &= C(\mathrm{I} + \mathrm{II} + \mathrm{III}).\end{aligned}\tag{2.13}$$

For the term I, we have

$$\begin{aligned}\mathrm{I} &\le \sum_{k,j}\beta|Q_j^k|_w\Phi(2^k) \le C\beta\sum_k \Phi(2^{k-1})|\Omega_k|_w\\ &\le C\beta\sum_k\int_{2^{k-1}}^{2^k}|\{T(f\mu) > \lambda\}|_w\,d\Phi(\lambda) = C\beta\int_{R^n}\Phi(T(f\mu))\,dw.\end{aligned}\tag{2.14}$$

Now we estimate the term II:

$$\begin{aligned}\mathrm{II} &\le \sum_{(k,j)\in F}|E_j^k|_w\Phi\left(\frac{2\sigma_j^k}{|E_j^k|_w}\right) \le \sum|E_j^k|_w\Phi\left(\frac{2\sigma_j^k}{\beta|E_j^k|_w}\right)\\ &\le C_\beta\sum|E_j^k|_w\Phi\left(\frac{1}{|Q_j^k|_w}\int_{3Q_j^k\setminus\Omega_{k+m}} fT(\chi_{E_j^k}w)\,d\mu\right),\end{aligned}\tag{2.15}$$

where $C_\beta = (2/\beta)^{p_\Phi}$ by use of (2.9).

Let (k, j) be fixed and let $\varepsilon = \varepsilon_{k,j} > 0$. On writing $R_j^k = 3Q_j^k\setminus\Omega_{k+m}$, we have

$$\frac{1}{|Q_j^k|_w}\int_{3Q_j^k\setminus\Omega_{k+m}} fT(\chi_{E_j^k}w)\,d\mu \le \frac{1}{\varepsilon_{k,j}|Q_j^k|_w}\|f\chi_{R_j^k}\|_{\Phi(\varepsilon\mu)}\|T(\chi_{Q_j^k}w)\|_{\Psi(\varepsilon\mu)}.$$

If $\varepsilon = \varepsilon_{k,j}$ is chosen such that $\int\Phi(f\chi_{R_j^k})\varepsilon_{k,j}\,d\mu = 1$ then $\|f\chi_{R_j^k}\|_{\Phi(\varepsilon\mu)} \le 1$, and from (1.11) it follows that

$$\|T(\chi_{Q_j^k}w)\|_{\Psi(\varepsilon\mu)} \le C\|\chi_{Q_j^k}\|_{\Psi(\varepsilon w)} \le C\Phi^{-1}\left(\frac{1}{\varepsilon_{k,j}|Q_j^k|_w}\right)\varepsilon_{k,j}|Q_j^k|_w.\tag{2.16}$$

Therefore, we obtain

$$\frac{1}{|Q_j^k|_w}\int_{R_j^k} fT(\chi_{E_j^k}w)\,d\mu \le C\Phi^{-1}\left(\frac{1}{\varepsilon_{k,j}|Q_j^k|_w}\right).$$

On substituting the last estimate into (2.15), it follows that

$$\begin{aligned}\mathrm{II} &\le C_\beta C\sum_{k,j}\frac{|E_j^k|_w}{\varepsilon_{k,j}|Q_j^k|_w} \le C\sum_{k,j}\int \Phi(f\chi_{R_j^k})\,d\mu \\ &\le C\int_{R^n}\Phi(f)\,d\mu.\end{aligned}\tag{2.17}$$

The last inequality holds since $\sum_{k,j}\chi_{R_j^k} \le C\sum_k \chi_{\Omega_k\setminus\Omega_{k+m}} \le C(m+1)$ by (2.11)(3).

To estimate the term III in (2.13), we need the following Φ-inequality for the dyadic maximal operator. For σ a positive Borel measure on R^n, define

$$M_\sigma^d f(x) = \sup_{x\in Q,\ \text{dyadic cube}}\frac{1}{|Q|_\sigma}\int_Q |f(y)|\,d\sigma(y)\quad\text{for } f\in L_{\text{loc}}(\sigma).$$

Then for any given Young's function Φ as in Theorem 2,

$$\int_{R^n}\Phi(M_\sigma^d f)\,d\sigma C\int_{R^n}\Phi(f)\,d\sigma\tag{2.18}$$

holds for all $f\in L_\Phi(\sigma)$.

Indeed, it is known that the operator M_σ^d is weak type $(1, 1)$ with the constant 1 with respect to the measure σ. It follows that

$$\begin{aligned}\int_{R^n}\Phi(M_\sigma^d f)\,d\sigma &\le C\int_0^\infty |\{M_\sigma^d f > 2\lambda\}|_\sigma \phi(\lambda)\,d\lambda \\ &\le C\int_0^\infty\left(\frac{1}{\lambda}\int_{\{|f|>\lambda\}}|f|\,d\sigma\right)\frac{\Phi(\lambda)}{\lambda}\,d\lambda \\ &= C\int_{R^n}\int_0^{|f|}\frac{\Phi(\lambda)}{\lambda^2}\,d\lambda|f|\,d\sigma \le C\int_{R^n}\Phi(|f|)\,d\sigma\end{aligned}$$

since

$$\int_0^u\frac{\Phi(\lambda)}{\lambda^2}\,d\lambda = \frac{1}{u}\int_0^1\frac{\Phi(tu)}{t^2}\,dt \le \frac{\Phi(u)}{u}\int_0^1\frac{dt}{t^{2-q_\Phi}} = C\frac{\Phi(u)}{u}.$$

Let $H_j^k = \{i\colon Q_i^{k+m}\cap 3Q_j^k \ne \varnothing\}$ and $L_j^k = \{s\colon Q_s^k\cap 3Q_j^k\ne\varnothing\}$. Then for given (k, j),

$$3Q_j^k\cap\Omega_{k+m}\subset\bigcup_{i\in H_j^k}Q_i^{k+m}\subset\bigcup_{s\in L_j^k}\ \bigcup_{i:Q_i^{k+m}\subset Q_s^k}Q_i^{k+m}.$$

Set $A_j^k = (1/|Q_j^k|_\mu)\int_{Q_j^k} f\,d\mu$. In [8] it has been obtained that

$$r_j^k \le C\sum_{i\in H_j^k}\left(\int_{Q_i^{k+m}}T(\chi_{E_j^k}w)\,d\mu\right)A_i^{k+m}.\tag{2.19}$$

Following the approach in [8], we shall prove that

$$\sum_{\substack{(k,j)\in G\\ k\geq N,\ k\equiv M (\mathrm{mod}\ m)}} |E_j^k|_w \Phi(2^k) \leq C\int_{R^n} \Phi(f)\,d\mu \tag{2.20}$$

with a constant C independent of the integers N and M, where $N \in (-\infty, \infty)$, $0 \leq M < m$. The indices (k, j) are restricted by this convention until the proof of (2.20) is completed.

As in [8], we select the "principle" cubes from $\{Q_j^k\}$. Let G_0 consist of those indices (k, j) for which Q_j^k is maximal. If G_n has been defined, for every index (t, u) in G_n we select the maximal cubes $Q_j^k \subset Q_u^t$ such that $A_j^k > 2A_u^t$. The indices of those cubes so selected form the set G_{n+1}. Define $\Gamma = \bigcup_{n=0}^{\infty} G_n$, and for each (k, j), define $P(Q_j^k)$ to be the smallest cube Q_u^t containing Q_j^k with $(t, u) \in \Gamma$. Then we have

$$\begin{aligned}&(1)\ P(Q_j^k) = Q_u^t \text{ implies } A_j^k \leq 2A_u^t;\\ &(2)\ Q_j^k \subset Q_u^t \text{ and } (k, j), (t, u) \in \Gamma \text{ imply } A_j^k > 2A_u^t.\end{aligned} \tag{2.21}$$

Observing that the cardinality of L_j^k is at most C and if $Q_i^{k+m} \subset Q_s^k$ with $(k+m, i) \notin \Gamma$ then $P(Q_i^{k+m}) = P(Q_s^k)$, we obtain

(2.22)

$$\begin{aligned}
\sum_{(k,j)\in G} |E_j^k|_w \Phi(2^k) &\leq C \sum |E_j^k|_w \Phi\left(\frac{r_j^k}{|Q_j^k|_w}\right) \quad \text{(by (2.12) and (2.8))}\\
&\leq C\sum |E_j^k|_w \Phi\left(\frac{1}{|Q_j^k|_w} \sum_{s\in L_j^k} \sum_{\substack{i: P(Q_i^{k+m}) = P(Q_s^k),\\ Q_i^{k+m}\subset Q_s^k}} \left[\left(\int_{Q_i^{k+m}} T(\chi_{Q_j^k} w)\,d\mu\right) A_i^{k+m}\right]\right)\\
&\quad + C\sum |E_j^k|_w \Phi\left(\frac{1}{|Q_j^k|_w} \sum_{i\in H_j^k : (k+m, j)\in\Gamma} \left[\left(\int_{Q_i^{k+m}} T(\chi_{Q_j^k} w)\,d\mu\right) A_i^{k+m}\right]\right)\\
&\qquad\qquad \text{(by (2.19))}\\
&\leq C\sum |E_j^k|_w \sum_{s\in L_j^k} \Phi\left(\frac{1}{|Q_j^k|_w} \sum_{\substack{i: P(Q_i^{k+m}) = P(Q_s^k),\\ Q_i^{k+m}\subset Q_s^k}} \left[\left(\int_{Q_i^{k+m}} T(\chi_{Q_j^k} w)\,d\mu\right) A_i^{k+m}\right]\right)\\
&\quad + C\sum |E_j^k|_w \Phi\left(\frac{1}{|Q_j^k|_w} \sum_{i\in H_j^k : (k+m, j)\in\Gamma} \left[\left(\int_{Q_i^{k+m}} T(\chi_{Q_j^k} w)\,d\mu\right) A_i^{k+m}\right]\right)\\
&= C(\mathrm{IV} + \mathrm{V}).
\end{aligned}$$

To deal with IV, we deduce the following estimate for fixed $(t, u) \in \Gamma$.

(2.23)

$$\begin{aligned}
&\sum_{k,j} \sum_{s \in L_j^k : P(Q_s^k) = Q_u^t} |E_j^k|_w \Phi \left(\frac{1}{|Q_j^k|_w} \sum_{\substack{i : P(Q_i^{k+m}) = P(Q_s^k), \\ Q_i^{k+m} \subset Q_s^k}} \left[\left(\int_{Q_i^{k+m}} T(\chi_{Q_j^k} w)\, d\mu \right) A_i^{k+m} \right] \right) \\
&\quad \le \sum_{k,j} \sum_{s \in L_j^k : P(Q_s^k) = Q_u^t} |E_j^k|_w \Phi \left(\frac{2A_u^t}{|Q_j^k|_w} \int_{Q_s^k} T(\chi_{Q_j^k} w)\, d\mu \right) \quad \text{(by (2.21)(1))} \\
&\quad \le C \sum_{k,j} \sum_{s \in L_j^k : P(Q_s^k) = Q_u^t} |E_j^k|_w \Phi \left(\frac{1}{|Q_j^k|_w} \int_{Q_j^k} T(A_u^t \chi_{Q_u^t} \mu)\, dw \right) \\
&\quad \le C \sum_{k,j} \int_{E_j^k} \Phi(M_w^d(T(A_u^t \chi_{Q_u^t} \mu)))\, dw \quad \text{(since the cardinality of } L_j^k \le C) \\
&\quad \le C \int \Phi(T(A_u^t \chi_{Q_u^t} \mu))\, dw \quad \text{(by (2.18))} \\
&\quad \le C \Phi(A_u^t) |Q_u^t|_\mu \quad \text{(by (1.10))}.
\end{aligned}$$

Summing (2.23) over $(t, u) \in \Gamma$ yields

$$\mathrm{IV} \le C \sum_{(t,u) \in \Gamma} \Phi(A_u^t) |Q_u^t|_\mu. \tag{2.24}$$

Now we estimate V. For every (k, j) let $\Gamma_j^k = \{i : i \in H_j^k,\ (k+m, i) \in \Gamma\}$ and $P_j^k = \bigcup_{i \in \Gamma_j^k} Q_i^{k+m}$. Then

(2.25)

$$\begin{aligned}
V &= \sum_{(k,j) \in G} |E_j^k|_w \Phi \left(\frac{1}{|Q_j^k|_w} \sum_{i \in \Gamma_j^k} \left[\left(\int_{Q_i^{k+m}} T(\chi_{Q_j^k} w)\, d\mu \right) A_i^{k+m} \right] \right) \\
&= \sum_{(k,j) \in G} |E_j^k|_w \Phi \left(\frac{1}{|Q_j^k|_w} \int_{P_j^k} T(\chi_{Q_j^k} w) \left(\sum_{i \in \Gamma_j^k} A_i^{k+m} \chi_{Q_i^{k+m}} \right) d\mu \right).
\end{aligned}$$

Let (k, j) be fixed and let $\varepsilon = \varepsilon_{k,j} > 0$. Using the Hölder inequality in the Orlicz spaces, we have

(2.26)

$$\begin{aligned}
&\frac{1}{|Q_j^k|_w} \int_{P_j^k} T(\chi_{Q_j^k} w) \left(\sum_{i \in \Gamma_j^k} A_i^{k+m} \chi_{Q_i^{k+m}} \right) d\mu \\
&\quad \le \frac{C}{\varepsilon |Q_j^k|_w} \|T(\chi_{Q_j^k} w)\|_{\Psi(\varepsilon\mu)} \left\| \sum_{i \in \Gamma_j^k} A_i^{k+m} \chi_{Q_i^{k+m}} \right\|_{\Phi(\varepsilon\mu)}.
\end{aligned}$$

If $\varepsilon_{k,j}$ is chosen such that $\int \Phi(\sum_{i\in\Gamma_j^k} A_i^{k+m}\chi_{Q_i^{k+m}})\varepsilon_{k,j}\,d\mu = 1$, then

$$\left\| \sum_{i\in\Gamma_j^k} A_i^{k+m}\chi_{Q_i^{k+m}} \right\|_{\Phi(\varepsilon\mu)} \le 1,$$

and by use of (2.16) and (2.26) we have

$$\Phi\left(\frac{1}{|Q_j^k|_w}\int_{P_j^k} T(\chi_{Q_j^k}w)\left(\sum_{i\in\Gamma_j^k} A_i^{k+m}\chi_{Q_i^{k+m}}\right)d\mu\right) \le \frac{C}{\varepsilon_{k,j}|Q_j^k|_w}.$$

Therefore it follows that

$$\begin{aligned} V &\le C\sum_{(k,j)\in G}\frac{|E_j^k|_w}{\varepsilon_{k,j}|Q_j^k|_w} \le C\sum_{(k,j)\in G}\frac{1}{\varepsilon_{k,j}} \\ &= C\sum_{(k,j)\in G}\sum_{i\in\Gamma_j^k}\Phi(A_i^{k+m})|Q_i^{k+m}|_\mu \le C\sum_{(t,u)\in\Gamma}\Phi(A_u^t)|Q_u^t|_\mu \end{aligned} \tag{2.27}$$

since any fixed Q_i^{k+m} occurs at most C times in the above sum (see [7, 8]).

From (2.9) and (2.21)(2) it follows that for any fixed x

$$\sum_{(t,u)\in\Gamma}\Phi(A_u^t)\chi_{Q_u^t}(x) \le C\sup_{x\in Q_u^t}\Phi(A_u^t)\sum_{k=0}^{\infty}\frac{1}{2^{kq_\Phi}} \le C\Phi(M_\mu^d f(x)).$$

Combining (2.22), (2.24), and (2.27) shows that the left side of (2.20) is bounded by

$$\begin{aligned} C\sum_{(t,u)\in\Gamma}\Phi(A_u^t)|Q_u^t|_\mu &\le C\int\sum_{(t,u)\in\Gamma}\Phi(A_u^t)\chi_{Q_u^t}(x)\,d\mu \\ &\le C\int\Phi(M_\mu^d f(x))\,d\mu \le C\int\Phi(f)\,d\mu \quad \text{(by (2.18))}. \end{aligned} \tag{2.28}$$

Let $N\to\infty$ in (2.20) and then sum over $M = 0, 1, \ldots, m-1$ to obtain

$$\mathrm{III} \le C\int\Phi(f)\,d\mu. \tag{2.29}$$

In (2.14) choose β so small that $C\beta < \frac{1}{2}$. It is easy to conclude (1.9) from (2.13), (2.14), (2.17), and (2.29). Hence for arbitrary $f \ge 0$ we obtain (1.9) by the monotone convergence theorem. This completes the proof of Theorem 2.

Proof of Theorem 3. The proof of the necessity is easy and we omit it. For the sufficiency, first, we prove the theorem for the relative dyadic maximal operator

$$Nf(x,t) = \sup\frac{1}{|Q|}\int_Q |f(y)|\,dy, \qquad x\in R^n,\ t\ge 0,$$

where the supremum is taken over the dyadic cubes in R^n containing x and having side length at least t. In addition, assuming the side length at most R, we denote the associated operator by N^R. Obviously, $N^R f(x,t) = 0$ for $t > R$ and $\lim N^R f(x,t) = Nf(x,t)$ $(R\to\infty)$.

We shall prove that (1.15) is true for N^R with constant C independent of R if (1.16) holds for all dyadic cubes. Once this is proved, a limit argument shows that (1.15) holds for N.

Suppose $f \geq 0$. For every integer k let $\Omega_k = \{(x, t): N^R(fw) > 2^k\}$. According to a decomposition lemma in [6], there exists a family $\{Q_j^k\}_{j\in J_k}$ of dyadic cubes such that

$$\text{(2.30)}\qquad \begin{aligned} &(1)\ (1/|Q_j^k|)\textstyle\int_{Q_j^k} fw > 2^k;\\ &(2)\ \text{the interiors of } Q_j^k \text{ are disjoint};\\ &(3)\ \Omega_k = \textstyle\bigcup_{j\in J_k} \widehat{Q}_j^k. \end{aligned}$$

Writing $E_j^k = \widehat{Q}_j^k \backslash \Omega_{k+1}$ we have

$$\text{(2.31)}\qquad \begin{aligned} \int_{R_+^{n+1}} \Phi(N^R(fw))\,d\mu &\leq C\sum_{k,j} \Phi(2^k)\mu(E_j^k)\\ &\leq C\sum \mu(E_j^k)\Phi\left(\frac{|Q_j^k|_w}{|Q_j^k|}\right)\Phi\left(\frac{1}{|Q_j^k|_w}\int_{Q_j^k} fw\right) \end{aligned}$$

by use of (2.30)(1) and the Δ'-condition. For every integer s let

$$\Gamma_s = \left\{(k, j): 2^s < \Phi\left(\frac{1}{|Q_j^k|_w}\int_{Q_j^k} fw\right) \leq 2^{s+1}\right\}, \qquad G_s = \bigcup_{(k,j)\in\Gamma_s} Q_j^k.$$

Since all the cubes in the doubly indexed family $\{Q_j^k: (k, j) \in \Gamma_s\}$ have side length $\leq R$, every cube will be contained in a maximal one. Let $\{Q_i\}_{i\in\Gamma_s'}$ be the subfamily formed by these maximal cubes. Then $G_s = \bigcup_{i\in\Gamma_s'} Q_i \subset \{x \in R^n: \Phi(M_w^d f(x)) > 2^s\}$. Therefore the right side of (2.31) is bounded by

$$\text{(2.32)}\qquad \begin{aligned} &C\sum_{s=-\infty}^{\infty}\sum_{i\in\Gamma_s'}\sum_{\substack{(k,j)\in\Gamma_s,\\ Q_j^k\subset Q_i}} \mu(E_j^k)\Phi\left(\frac{|Q_j^k|_w}{|Q_j^k|}\right)\Phi\left(\frac{1}{|Q_j^k|_w}\int_{Q_j^k} fw\right)\\ &\leq C\sum_{s=-\infty}^{\infty}\sum_{i\in\Gamma_s'}\sum_{\substack{(k,j)\in\Gamma_s,\\ Q_j^k\subset Q_i}} 2^{s+1}\int_{E_j^k} \Phi(N(\chi_{Q_j^k} w)(x, t))\,d\mu(x, t)\\ &\leq C\sum_s 2^{s+1}\sum_{i\in\Gamma_s'}\int_{\widehat{Q}_i} \Phi(N(\chi_{Q_i} w)(x, t))\,d\mu(x, t)\\ &\leq C\sum_s 2^{s+1}\sum_{i\in\Gamma_s'} |Q_i|_w \quad \text{(by (1.16))}\\ &\leq C\sum_s 2^{s+1}|\{x \in R^n: \Phi(M_w^d(f(x)) > 2^s\}|_w\\ &\leq C\int \Phi(M_w^d f)w\,dx \leq C\int \Phi(f)w\,dx \quad \text{(by (2.18))}. \end{aligned}$$

The rest of the proof follows easily from the following variation of Sawyer's lemma that appeared in [6].

Lemma. *Define for each* $y \in R^n$

$$^{y}Nf(x, t) = \sup \frac{1}{|Q|} \int_Q |f(u)|\, du,$$

the supremum being taken in all cubes Q *with* $x \in Q$, *side length more that* t, *and such that the set* $Q - y = \{x - y : x \in Q\}$ *is a dyadic cube. Then*

$$M^{2^k} f(x, t) \le \frac{2^{3n+1}}{|Q(0, 2^{k+2})|} \int_{Q(0, 2^{k+2})} {}^{y}Nf(x, t)\, dy,$$

where $Q(0, 2^{k+2}) = [-2^{k+2}, 2^{k+2}]^n$ *and by* M^r *we mean the maximal operator obtained by considering cubes with side length less than* r. *Then we have*

$$\begin{aligned}
&\int_{R_+^{n+1}} \Phi(M^{2^k}(fw)(x, t))\, d\mu(x, t) \\
&\quad \le C \int_{R_+^{n+1}} \Phi\left(\frac{2^{3n+1}}{|Q(0, 2^{k+2})|} \int_{Q(0, 2^{k+2})} {}^{y}Nf(x, t)\, dy \right) d\mu \\
&\quad \le \frac{C}{|Q(0, 2^{k+2})|} \int_{Q(0, 2^{k+2})} \int_{R_+^{n+1}} \Phi({}^{y}N(fw)(x, t))\, d\mu(x, t)\, dy \\
&\quad \le C \int \Phi(f) w\, dx.
\end{aligned}$$

By letting $k \to \infty$ *we conclude* (1.15). *Theorem* 3 *is proved.*

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