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NEF COTANGENT BUNDLES OF BRANCHED COVERINGS

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ABSTRACT. In a previous paper we characterized those $H(\Lambda, n)$ (compact complex surfaces constructed by Hirzebruch) that have nef cotangent bundle. In this article we extend the methods to study more general branched coverings with regard to nefness of their cotangent bundles.

1. INTRODUCTION

We first develop the notation. Let $\Lambda = \{L_1, \dots, L_k\}$ be an arrangement of k lines in $\mathbf{P}^2 = \mathbf{P}^2(\mathbf{C})$. For $p \in \mathbf{P}^2$, r_p is the number of lines in Λ containing p. $\mathbf{BP}^2$ is the blowup of $\mathbf{P}^2$ at all p with $r_p \geq 3$. Let L'_j denote the proper transform of L_j in $\mathbf{BP}^2$ and $\Lambda' = \bigcup\{L'_j\}$. One defines t_j by $t_j =$ cardinality$\{p \in \mathbf{P}^2 | r_p = j\}$. Let E_i denote the exceptional curve in $\mathbf{BP}^2$ over p_i with $r_{p_i} \geq 3$, and let $\pi_i \colon \mathbf{BP}^2 \to E_i$ be projection. For $\pi \colon X \to \mathbf{BP}^2$ a branched covering, $s_i \colon X \to SC_i$ will be the Stein factorization of $\pi_i \circ \pi \colon X \to E_i$.

For any vector bundle E over a base manifold M, the projectivization $\mathbf{P}(E)$ is a fiber bundle over M, with fiber $\mathbf{P}_q(E)$ over $q \in M$ given by $\mathbf{P}_q(E) \approx (E_q^* \setminus 0)/\mathbf{C}^*$. There is a tautological line bundle ξ_E over $\mathbf{P}(E)$ satisfying (i) $\xi_E|_{\mathbf{P}_q(E)} \approx O(1)_{\mathbf{P}_q(E)} \forall q \in M$ and (ii) the projection $\rho_E \colon \mathbf{P}(E) \to M$ gives $\rho_{E*}(\xi_E) \approx E$. In the case that $E = T^*M$ we will denote $\rho_E = \rho_{T^*M}$ simply by ρ.

Recall that the vector bundle E is nef if ξ_E over $\mathbf{P}(E)$ is nef, that is, $c_1(\xi_E) \cdot C \geq 0$ for all effective curves C in $\mathbf{P}(E)$.

In what follows, $g(C)$ and $e(C)$ will denote the genus and euler number, respectively, of a curve C. For further development see [4, 5, 1].

The problem approached here will be: given a branched covering $\pi \colon X \to Y$ of compact complex surfaces, find conditions on Y along with the ramification set of π in X sufficient to guarantee that T^*X is nef. Cases handled include: (i) $Y = \mathbf{BP}^2$ and (ii) T^*Y is nef.

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2. Nef cotangent bundles

In this section we prove the main theorems, including the following working theorem.

Theorem 1. *Let X and Z be compact complex manifolds with $\dim_{\mathbf{C}} X = 2$, $\dim_{\mathbf{C}} Z \geq 2$, and with Z having nef cotangent bundle. Let $\pi\colon X \to Z$ be a holomorphic mapping with $X' := \{x \in X \mid \pi_*|_x$ is not of maximal rank$\}$. Assume that $X' = \bigcup B_j \cup q_1 \cup q_2 \cup \cdots \cup q_N$, where the B_j are irreducible, smooth curves and the $q_1, q_2, \ldots, q_N$ are points. Then X has nef cotangent bundle if $B_j \cdot B_j \leq 0$ and $e(B_j) \leq 0$ for each B_j in $\bigcup B_j$.*

Proof. Assume that $B_j \cdot B_j \leq 0$ and $e(B_j) \leq 0$ for each B_j in $\bigcup B_j$. We prove nefness of T^*X.

The mapping $\pi\colon X \to Z$ induces $\pi_*\colon TX \to TZ$, which in turn induces $\Pi\colon \mathbf{P}(T^*X) \to \mathbf{P}(T^*Z)$. Let ξ_1 be the tautological bundle over $\mathbf{P}(T^*X)$, and let ξ_2 be the tautological bundle over $\mathbf{P}(T^*Z)$. Π has indeterminacy set that is contained in $\rho^{-1}(A)$ where $\rho\colon \mathbf{P}(T^*X) \to X$ is projection and $A = \bigcup B_j \cup q_1 \cup \cdots \cup q_N$.

Let C be an effective irreducible curve in $\mathbf{P}(T^*X)$. For T^*X to be nef we show that $C \cdot \xi_1^{-1} \leq 0$. This is accomplished in three cases.

Case 1. Suppose that $\rho^{-1}(A) \not\supset C$. Let $\nu\colon \eta C \to C$ be the normalization of C. In this setting $\Pi \circ \nu\colon C \to \mathbf{P}(T^*Z)$ extends to be well defined and, by Lemma 2, we have that $(\Pi \circ \nu)^*(\xi_2^{-1}) \cong \nu^*(\xi_1^{-1}) + D$ where D is an effective divisor on ηC. So

$$\begin{aligned} C \cdot \xi_1^{-1} &= \eta C \cdot \nu^*(\xi_1^{-1}) = \eta C \cdot ((\Pi \circ \nu)^*(\xi_2^{-1}) - D) \\ &= \deg(\Pi \circ \nu)(\Pi \circ \nu)(C) \cdot \xi_2^{-1} - \eta C \cdot D \leq 0. \end{aligned}$$

The last inequality follows since $\Pi \circ \nu(C)$ is a curve in $\mathbf{P}(T^*Z)$, ξ_2 is nef, and D is effective on ηC.

Case 2. Suppose that C is a fiber of ρ (and therefore $C \cong \mathbf{P}^1$). Then $C \cdot \xi_1^{-1} = -1$ since $\xi_1|_C \cong O(1)$.

Case 3. Suppose C is contained in $\rho^{-1}(A)$ but is not a fiber of ρ. Then $\rho(C) = B_j$ for some j. For $\nu\colon \eta C \to C$ the normalization of C, one has that $\rho \circ \nu\colon \eta C \to \rho(C)$. One has the vector bundle maps

$$0 \to \nu^*(\xi_1^{-1}|_C) \to \nu^*\rho^*(TX|_{\rho(C)})$$

and

$$0 \to \nu^*\rho^* T\rho(C) \to \nu^*\rho^*(TX|_{\rho(C)}) \to \nu^*\rho^* N\rho(C) \to 0.$$

Hence one of the sequence of sheaves (2.1) or (2.2) must be valid.

$$0 \to \nu^*\xi_1^{-1} \to \nu^*\rho^* T\rho(C) \to Z_1 \to 0, \tag{2.1}$$

$$0 \to \nu^*\xi_1^{-1} \to \nu^*\rho^* N\rho(C) \to Z_2 \to 0 \tag{2.2}$$

where Z_1 and Z_2 are sheaves with finite support on ηC. By letting $M_1 = \nu^*\rho^* T\rho(C)$ and $M_2 = \nu^*\rho^* N\rho(C)$, we rewrite (2.1) and (2.2) as

$$0 \to \nu^*\xi_1^{-1} \to M_i \to Z_i \to 0. \tag{2.3}$$

By utilizing the long exact sequence associated to (2.3), along with Riemann-Roch, one concludes that

$$c_1(\xi_1^{-1}) \cdot C = c_1(\nu^* \xi_1^{-1}) \cdot \eta C \leq c_1(M_i) \cdot \eta C = \begin{cases} \deg(\rho \circ \nu) c_1(T\rho(C)) \cdot \rho(C), \\ \deg(\rho \circ \nu) c_1(N\rho(C)) \cdot \rho(C). \end{cases}$$

Now $c_1(T\rho(C)) \cdot \rho(C) = e(\rho(C)) = e(B_j) \leq 0$ by our hypothesis. And similarly we have that $c_1(N(\rho(C)) \cdot \rho(C) = \rho(C) \cdot \rho(C) = B_j \cdot B_j \leq 0$ by hypothesis. Therefore, in all cases $\xi_1^{-1} \cdot C \leq 0$. So $\xi_1 \cdot C \geq 0$ and ξ_1 is nef, giving that T^*X is nef. □

In the setting of Theorem 1 we prove

Lemma 2. *For C contained in $\mathbf{P}(T^*X)$ with C not contained in $\rho^{-1}(A)$, let $\nu \colon \eta C \to C$ be normalization. Then*

$$(\Pi \circ \nu)^*(\xi_2^{-1}) \cong \nu^* \xi_1^{-1} + D$$

where D is an effective divisor on ηC.

Proof. First note that if $g \colon L_1 \to L_2$ is a holomorphic mapping of complex line bundles over a curve (which we take to be) ηC, then $L_2 \cong L_1 + D_g$ where D_g is the effective divisor on ηC induced by the vanishing of the mapping g. Next we let $L_1 = \nu^* \xi_1^{-1}$ and $L_2 = (\Pi \circ \nu)^*(\xi_2^{-1})$ where $\Pi \circ \nu$ extends to be well defined on ηC as C is not contained in $\rho^{-1}(A)$. Observe that $\pi_* \colon TX \to TZ$ induces the mapping $\pi_* \colon \xi_1^{-1} \to \xi_2^{-1}$, which in turn induces $\pi_* \colon \nu^* \xi_1^{-1} \to (\Pi \circ \nu)^*(\xi_2^{-1})$. Finally we take g to be the mapping $\pi_* \colon \nu^* \xi_1^{-1} \to (\Pi \circ \nu)^*(\xi_2^{-1})$, giving the lemma with $D := D_g$. □

Theorem 3. *Let Λ be an arrangement of $k \geq 3$ lines in $\mathbf{P}^2$ with $t_k = 0$. Assume there are at least two points p_i, $i = 1, 2$, with $r_{p_i} \geq 3$. Let $\mathbf{BP}^2$ be the blowup of $\mathbf{P}^2$ at each point p_j with $r_{p_j} \geq 3$, and let E_j be the exceptional curve over p_j with $\pi_j \colon \mathbf{BP}^2 \to E_j$ projection. Let L be the line containing p_1 and p_2 in $\mathbf{P}^2$, and let L' be its proper transform in $\mathbf{BP}^2$.*

Let $\pi \colon X \to \mathbf{BP}^2$ be any branched covering of $\mathbf{BP}^2$ with branch locus contained in $\Lambda' \cup E_1 \cup \cdots \cup E_n \cup L'$ where $\pi^{-1}(\Lambda' \cup E_1 \cup \cdots \cup E_n \cup L')$ has irreducible components that are smooth. If $\pi \colon X \to \mathbf{BP}^2$ satisfies

(1) *For each $L_\alpha \in \Lambda \cup \{L\}$, $e(\widehat{L}_\alpha) \leq 0$, and $\widehat{L}_\alpha \cdot \widehat{L}_\alpha \leq 0$ for each irreducible component $\widehat{L}_\alpha$ of $\pi^{-1}(L'_\alpha)$;*
(2) *$e(C_j) \leq 0$ for each irreducible component C_j of $\pi^{-1}(E_j)$, $j = 1, \ldots, n$;*
(3) *$g(SC_i) \geq 1$ for $i = 1, 2$ where $s_i \colon X \to SC_i$ and $\rho_i \colon SC_i \to E_i$ is the Stein factorization of $\pi_i \circ \pi \colon X \to E_i$;*

then T^*X is nef.

Proof. Define $Z = SC_1 \times SC_2$ and construct the mapping $s \colon X \to Z$ given by $s = s_1 \times s_2$ where $s_i \colon X \to SC_i$ is the Stein factorization of $\pi_i \circ \pi \colon X \to E_i$. Then since $(\pi_1 \times \pi_2) \circ \pi = (\rho_1 \times \rho_2) \circ (s_1 \times s_2)$ and $\pi_1 \times \pi_2 \colon \mathbf{BP}^2 \to E_1 \times E_2$ is biholomorphic on $\mathbf{BP}^2 \backslash \bigcup E_1 \cup \cdots \cup E_n \cup L'$, the singular set of $s_1 \times s_2$ is contained in the set $\pi^{-1}(\Lambda' \cup E_1 \cup \cdots \cup E_n \cup L') = \bigcup \widehat{L}_j \cup \pi^{-1}(E_1) \cup \cdots \cup \pi^{-1(}(E_n) \cup \pi^{-1}(L')$.

By hypotheses (1) and (2), $e(B_j) \leq 0$ for each B_j that is contained in the set $\bigcup \widehat{L}_j \cup \pi^{-1}(E_1) \cup \cdots \cup \pi^{-1}(E_n) \cup \pi^{-1}(L')$. (1) gives that $B_j \cdot B_j \leq 0$ for B_j contained in the set $\bigcup \widehat{L}_j \cup \pi^{-1}(L')$. If B_j is contained in $\pi^{-1}(E_1) \cup \cdots \cup \pi^{-1}(E_n)$

then $B_j \cdot B_j \leq 0$, since if n_j is the branching order of π along B_j then $B_j \cdot B_j \leq (1/n_j)\deg(\pi|_{B_j})E_j^2 < 0$. Now (3) gives that T^*Z is nef. Apply Theorem 1 to the mapping $s\colon X \to Z$ to conclude that T^*X is nef. □

Remark. One can drop the extra hypotheses on L (the line of collinearity containing p_1 and p_2) of Theorem 3 and still conclude that T^*X is nef, provided that there is a third point p_3, noncollinear with p_1 and p_2, also satisfying $g(SC_3) \geq 1$ where $s_3\colon X \to SC_3$ is the Stein factorization of $\pi_3 \circ \pi\colon X \to E_3$. The proof is similar to that of Theorem 3 if one lets s be the mapping $s_1 \times s_2 \times s_3 : X \to SC_1 \times SC_2 \times SC_3$.

Theorem 4. *Let X and Y be compact complex surfaces with Y having nef cotangent bundle, and let $\pi\colon X \to Y$ be a branched covering with ramification set $\bigcup B_j$ in X, where we assume that the B_j are irreducible and smooth. Let $\bigcup C_\alpha$ be the branch locus of π in Y. Then*

$$C_\alpha \cdot C_\alpha \leq 0 \textit{ for each } C_\alpha \textit{ in the branch locus in } Y$$
$$\Rightarrow B_j \cdot B_j \leq 0 \textit{ for each } B_j \textit{ in the ramification set in } X$$
$$\Rightarrow X \textit{ has nef cotangent bundle}.$$

Proof. Assume that $C_\alpha \cdot C_\alpha \leq 0$ for each C_α in the branch locus in Y. Let $\pi^*(C_\alpha) = \sum_k n_{\alpha k} B_{\alpha k}$. Then

$$\begin{aligned} B_{\alpha j} \cdot \pi^*(C_\alpha) &= B_{\alpha j} \cdot \sum_k n_{\alpha k} B_{\alpha k} = n_{\alpha j} B_{\alpha j} \cdot B_{\alpha j} + B_{\alpha j} \cdot \sum_{k \neq j} n_{\alpha k} B_{\alpha k} \\ &= \deg(\pi|_{B_{\alpha j}}) C_\alpha \cdot C_\alpha \end{aligned}$$

whence

$$B_{\alpha j} \cdot B_{\alpha j} = n_{\alpha j}^{-1} \deg(\pi|_{B_{\alpha j}}) C_\alpha \cdot C_\alpha - n_{\alpha j}^{-1} B_{\alpha j} \cdot \sum_{k \neq j} n_{\alpha k} B_{\alpha k} \leq 0, \tag{2.4}$$

giving the first implication.

Assume that $B_j \cdot B_j \leq 0$ for each B_j in the ramification set. We prove nefness of T^*X. Observe that for each B_j in the ramification set $e(B_j) \leq 0$, otherwise $e(\pi(B_j)) > 0$ in Y, which contradicts that T^*Y is nef. Apply Theorem 1 to $\pi\colon X \to Y$ to conclude that T^*X is nef. □

Next we extend our viewpoint to arrangements of pencils of curves.

Definition. Let P_i be a pencil of curves in $\mathbf{P}^2$ and $\Lambda_i = \{C_{ij} | j = 1, \ldots, k_i\}$ be an arrangement of smooth curves in P_i with each C_{ij} intersecting each C_{iJ} transversely (only) in the base locus (consisting of isolated points). We label the base locus of P_i by $\mathrm{BL}P_i$.

(1) An *arrangement of pencils* Λ is defined to be $\Lambda = \bigcup \Lambda_i$ with each Λ_i as above.

(2) A *general arrangement of pencils* is defined to be an arrangement of pencils $\Lambda = \bigcup \Lambda_i$ satisfying:

(a) $\mathrm{BL}P_i \cap \mathrm{BL}P_j = \varnothing$ if $i \neq j$.

(b) After blowup of all points in $\bigcup \mathrm{BL}P_i$ to get $\mathbf{BP}^2$ and letting EC be the resulting set of exceptional curves and Λ' be the arrangement of proper transforms C'_{ij} of the curves C_{ij} in Λ, we have that $EC \cup \Lambda'$ consists of smooth curves meeting transversely in normal crossings.

(c) For each i and each j, C'_{ij} is a nonsingular fiber of π_i.

(d) There are $k \geq 3$ pencils, say $P_1, P_2, \ldots, P_k$, with $\Lambda \supset \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_k$ satisfying that $\pi_1 \times \pi_2 \times \cdots \times \pi_k \colon \mathbf{BP^2} \to \mathbf{P^1} \times \mathbf{P^2} \times \cdots \mathbf{P^1}$ is of maximal rank 2 on $\mathbf{BP^2} \setminus EC$ except possibly at finitely many points $q_1, q_2, \ldots, q_N$. Here each $\pi_i \colon \mathbf{BP^2} \to \mathbf{P^1}$ is the holomorphic mapping associated to the pencil P_i.

(e) For all i, no point of C_{ij} lies in the base locus of P_I for $I \neq i$.

Theorem 5. *Let Λ be a general arrangement of pencils in $\mathbf{P^2}$. Blowup all points in the base loci to obtain $\mathbf{BP^2}$. Let $\pi \colon X \to \mathbf{BP^2}$ be any branched covering of $\mathbf{BP^2}$ with branch locus in $\mathbf{BP^2}$ contained in $EC \cup \Lambda'$ and ramification set in $\pi^{-1}(EC \cup \Lambda')$ having irreducible components that are smooth.*

Let $\pi_i \colon \mathbf{BP^2} \to \mathbf{P^1}$ be the natural holomorphic projection associated to each pencil P_i. Let $s_i \colon X \to SC_i$ and $\rho_i \colon SC_i \to \mathbf{P^1}$ be the Stein factorization of $\pi_i \circ \pi \colon X \to \mathbf{P^1}$. Then T^*X is nef if:

(1) *For the projections $\pi_1, \ldots, \pi_k$ with $\pi_1 \times \pi_2 \times \cdots \times \pi_k \colon \mathbf{BP^2} \to \mathbf{P^1} \times \mathbf{P^1} \times \cdots \times \mathbf{P^1}$ of maximal rank on $\mathbf{BP^2} \setminus \{EC \cup$ finitely many points$\}$, we have that the euler numbers $e(SC_i) \leq 0$ for $i = 1, \ldots, k$.*

(2) *For each $C_j \in EC \cup \Lambda'$ and each irreducible component $\widehat{C}_j$ of $\pi^{-1}(C_j)$ one has that $e(\widehat{C}_j) \leq 0$ and $\widehat{C}_j \cdot \widehat{C}_j \leq 0$.*

Proof. Define $Z = SC_1 \times SC_2 \times \cdots \times SC_k$ and construct the mapping $s \colon X \to Z$ given by $s = s_1 \times s_2 \times \cdots \times s_k$ where $s_i \colon X \to SC_i$ is the Stein factorization of $\pi_i \circ \pi \colon X \to \mathbf{P^1}$. Then since $(\pi_1 \times \pi_2 \times \cdots \times \pi_k) \circ \pi = (\rho_1 \times \rho_2 \times \cdots \times \rho_k) \circ (s_1 \times s_2 \times \cdots \times s_k)$, the singular set of $s_1 \times s_2 \times \cdots \times s_k$ is contained in the singular set of $(\pi_1 \times \pi_2 \times \cdots \times \pi_k) \circ \pi \cup \pi^{-1}(EC)$. Therefore the singular set of $s_1 \times s_2 \times \cdots \times s_k$ is contained in $\pi^{-1}(q_1 \cup \cdots \cup q_N) \cup (\bigcup \widehat{C}_j)$. (1) gives that T^*Z is nef and (2) allows the application of Theorem 1 to conclude that T^*X is nef. □

Remark. See Example 4.4 for a construction of a class of surfaces to which we apply Theorem 5.

3. Applications

As an application of Theorem 3 we study galois branched coverings. Recall that a branched covering $\pi \colon X \to Y$ is galois if the deck transformations act transitively on fibers of π.

We begin by setting up the notation. Let Λ be an arrangement of lines in $\mathbf{P^2}$ with at least 2 points p_1 and p_2 having $r_{p_i} \geq 3$ for $i = 1, 2$. $\mathbf{BP^2}$ is the blowup of $\mathbf{P^2}$ at each point p_j with $r_{p_j} \geq 3$, and let E_j be the exceptional curve over p_j. Let L be the line containing p_1 and p_2 in $\mathbf{P^2}$, and let L' be its proper transform in $\mathbf{BP^2}$.

Let $\pi \colon X \to \mathbf{BP^2}$ be a galois branched covering that is locally of form $(u, v) \to (u^n, v^m)$ with branch locus in $\mathbf{BP^2}$ contained in $\Lambda' \cup (\bigcup E_j)$. The line $L'_i \in \Lambda'$ is assigned branching order n_i, where n_i is the branching order for each component $\widehat{L}'_i$ of $\pi^{-1}(L'_i)$. E_j is similarly assigned branching order m_j, where m_j is the branching order for each component $\widehat{E}_j$ of $\pi^{-1}(E_j)$.

Let the five parameter classes be given as follows:

(i) α parametrizes $L'_\alpha \in \Lambda' \cup \{L'\}$;

(ii) i parametrizes $L'_i \in \Lambda'$;

(iii) j parametrizes $E_j \in EC :=$ the set of exceptional curves in $\mathbf{BP}^2$;

(iv) given α, β parametrizes $p_\beta \in PL_\alpha$ where $PL_\alpha := \{p_\beta \in L'_\alpha | \exists$ two lines $L'_1, L'_2 \in \Lambda' \setminus L'_\alpha$ with $L'_1 \cap L'_2 = p_\beta\}$;

(v) given α, γ parametrizes $p_\gamma \in QL_\alpha$ where

$$QL_\alpha := \{p_\gamma \in L'_\alpha | \exists i \text{ with } i \neq \alpha \text{ with } p_\gamma \in L'_i \cap L'_\alpha \text{ and } p_\gamma \notin PL_\alpha\}.$$

For $p_\gamma \in QL_\alpha$ with $p_\gamma = L'_i \cap L'_\alpha$ we let $n_\gamma := n_i$. For $p_\beta \in PL_\alpha$ with $p_\beta = L'_i \cap L'_I$ let

$$M_\beta := \text{the least common multiple}(n_i, n_I)$$

where n_i and n_I are the respective branching orders of L'_i and L'_I.

Theorem 6. *Let Λ be an arrangement of lines in $\mathbf{P}^2$ with at least two points p_1 and p_2 having $r_{p_i} \geq 3$ for $i = 1, 2$. Let L be the line containing p_1 and p_2 in $\mathbf{P}^2$, and let L' be its proper transform in $\mathbf{BP}^2$. Let $\pi: X \to \mathbf{BP}^2$ be a galois branched covering that is locally of form $(u, v) \to (u^n, v^m)$ with branch locus in $\mathbf{BP}^2$ contained in $\Lambda' \cup (\bigcup E_j)$ and with $\pi^{-1}(\Lambda' \cup (\bigcup E_j) \cup L')$ having irreducible components that are smooth. Then, in the notation of the above paragraphs, T^*X is nef if*

$$(1)\ 2 - \sum_{p_\gamma \in QL_\alpha}\left(1 - \frac{1}{n_\gamma}\right) - \sum_{p_\beta \in PL_\alpha}\left(1 - \frac{1}{M_\beta}\right) - \sum_j \left(1 - \frac{1}{m_j}\right) E_j \cdot L'_\alpha \leq 0$$

for each $L_a \in \Lambda \cup \{L\}$.

(2) $2 - \sum_i (1 - 1/n_i) E_j \cdot L'_i \leq 0$ *for each E_j an exceptional curve.*

(3) *Each L_j in Λ is blown up at least once.*

(4) *$g(SC_i)$ is positive for $i = 1, 2$ where $s_i: X \to SC_i$ is the Stein factorization of $\pi_i \circ \pi: X \to \mathbf{P}^1$.*

Proof. This is a direct result of Theorem 3 after one computes that

$$e(\widehat{L}'_\alpha) = (\deg \pi|_{\widehat{L}'_\alpha}) \left(2 - \sum_{p_\gamma \in QL_\alpha}\left(1 - \frac{1}{n_\gamma}\right) - \sum_{p_\beta \in PL_\alpha}\left(1 - \frac{1}{M_\beta}\right) - \sum_j \left(1 - \frac{1}{m_j}\right) E_j \cdot L'_\alpha\right),$$

$$e(\widehat{E}_j) = (\deg \pi|_{\widehat{E}_j}) \left(2 - \sum_i \left(1 - \frac{1}{n_i}\right) E_j \cdot L'_i\right),$$

and

$$\widehat{L}'_\alpha \cdot \widehat{L}'_\alpha \leq \frac{1}{n_\alpha} \det(\pi|_{\widehat{L}'_\alpha}) L'_\alpha \cdot L'_\alpha \quad \text{as in (2.4)}.$$

Here $n_\alpha = 1$ if $L_\alpha = L$ and $L \notin \Lambda$. See Höfer's dissertation [3] for similar calculations. The two euler numbers are negative by (1) and (2) of the theorem. $L'_\alpha \cdot L'_\alpha \leq 0$ by (3) and the fact that L' has been blown up at least twice. □

Remark. There are many situations where hypothesis (4) of Theorem 6 holds naturally. See Example 4.1.

We give similarly an application of Theorem 4 in the galois setting.

Theorem 7. *Let T^*Y be nef and let $\pi: X \to Y$ be a galois branched covering having ramification set $\bigcup B_j$ in X and branch locus $\bigcup C_\alpha$ in Y. We assume that the B_j are smooth and meet transversely in at most normal crossings and that the C_α are smooth. We further assume that π can be locally represented by coordinate charts of the form $(u, \nu) \to (u^n, \nu^m)$. Then*

$$C_\alpha \cdot C_\alpha \leq 0 \text{ for each } C_\alpha \text{ in the branch locus in } Y$$
$$\Leftrightarrow B_j \cdot B_j \leq 0 \text{ for each } B_j \text{ in the ramification set in } X$$
$$\Leftrightarrow X \text{ has nef cotangent bundle.}$$

Proof. Under the assumptions that π is galois and locally of form $(u, v) \to (u^n, v^m)$ and the C_α are smooth, it cannot happen that $B_{\alpha j}$ meets $B_{\alpha k}$ for $j \neq k$ where $B_{\alpha j}$ and $B_{\alpha k}$ are irreducible components of $\pi^*(C_\alpha)$. (If $p \in B_{\alpha j} \cap B_{\alpha k}$ and (u, v) is a coordinate chart centered at p with π of form $(u, v) \to (u^n, v^m) =: (A, B)$, then the only local branching occurs at $u = 0$ and $v = 0$. Thus $B_{\alpha j}$ is given locally by, say, $u = 0$ and then $B_{\alpha k}$ is given by $v = 0$ (and on our chart $n = m$). Then $\pi(\{u = 0\} \cup \{v = 0\}) = \{A = 0\} \cup \{B = 0\}$, which contradicts that C_α is smooth.) Hence (2.4) gives that $B_{\alpha j} \cdot B_{\alpha j} = n_{\alpha j}^{-1} \deg(\pi|_{B_{\alpha j}}) C_\alpha \cdot C_\alpha$. So in this setting $B_{\alpha j} \cdot B_{\alpha j} \leq 0$ if and only if $C_\alpha \cdot C_\alpha \leq 0$, giving the first equivalence.

Theorem 4 gives that if $B_j \cdot B_j \leq 0$ for all curves B_j in the ramification set then T^*X is nef. It only remains to show the converse. We adapt a splitting lemma of Sommese [4]: if B_j is smooth and contained in the ramification set of π, then there is a splitting $TB_j \oplus NB_j \approx TX|_{B_j}$. The splitting is obtained by utilizing the natural sequence $0 \to TB_j \to TX|_{B_j} \to NB_j \to 0$ and producing a sub line bundle L of $TX|_{B_j}$ that projects onto NB_j. L is the unique line bundle contained in the annihilator of $\pi^* dA = nu^{n-1} du$ and $\pi^* dB = mv^{m-1} dv$. Here (u, v) are local coordinates in X with B_j being given locally by $u = 0$, and (A, B) are local coordinates in Y with $A = u^n$ and $B = v^m$. Once we have this splitting, B_j and NB_j determine a curve C in $\mathbf{P}(T^*X)$ and $\xi_1^{-1} \cdot C = NB_j \cdot B_j = B_j \cdot B_j \leq 0$ since $\xi_1 \cdot C$ is nonnegative by our assumption that T^*X is nef. □

4. Examples

Example 4.1. In [5], $H(\Lambda, n)$ with $k \geq 3$ was shown to have nef cotangent bundle iff

(a) For each $L \in \Lambda$, L is blown up at least once (in obtaining $\mathbf{BP}^2$).

(b) If $n = 2$ then $t_3 = \#\{p | r_p = 3\} = 0$.

(c) For all $L \in \mathbf{P}^2$, $\delta(L) \neq 2$, and if $n = 2$ then $\delta(L) \neq 3$ where $\delta(L)$ is defined as the cardinality of the branch locus in L' of $\pi|_{\pi^{-1}(L')}: \pi^{-1}(L') \to L'$.

Theorem 6 gives the sufficiency of (a), (b), and (c) for nefness of $T^*H(\Lambda, n)$. This follows since $H(\Lambda, n)$ is a galois branched covering of $\mathbf{BP}^2$, and one has in this setting that all branching orders are n. Thus

$$2 - \sum_i \left(1 - \frac{1}{n_i}\right) E_j \cdot L_i' = 2 - \left(1 - \frac{1}{n}\right) \left(\sum_i E_j \cdot L_i'\right) = 2 - \left(1 - \frac{1}{n}\right) r_{p_j},$$

and so Theorem 6(2) holds iff $2-(1-1/n)r_{p_j} \leq 0$ iff (b) holds. Similarly one has that

$$\begin{aligned} 2 - &\sum_{p_\gamma \in QL_\alpha}\left(1-\frac{1}{n_\gamma}\right) - \sum_{p_\beta \in PL_\alpha}\left(1-\frac{1}{M_\beta}\right) - \sum_j\left(1-\frac{1}{m_j}\right)E_j\cdot L'_\alpha \\ &= 2 - \sum_{p_\gamma \in QL_\alpha}\left(1-\frac{1}{n}\right) - \sum_{p_\beta \in PL_\alpha}\left(1-\frac{1}{n}\right) - \sum_j\left(1-\frac{1}{n}\right)E_j\cdot L'_\alpha \\ &= 2 - \left(1-\frac{1}{n}\right)\left(\# QL_\alpha + \# PL_\alpha + \sum_j E_j\cdot L'_\alpha\right) = 2 - \left(1-\frac{1}{n}\right)\delta(L_\alpha), \end{aligned}$$

and so Theorem 6(1) holds iff $2-(1-1/n)\delta(L_\alpha) \leq 0$ iff (under the assumption that (b) holds) (c) holds. Clearly Theorem 6(3) is the same condition as (a) above. Finally Theorem 6(4) holds automatically by the fact that each component of $\pi^{-1}(E_j)$ is a section of $s_j\colon H(\Lambda, n) \to SC_j$ (see [4]) so $g(\widehat{E}_j) = s(SC_j) > 0$ by (b).

We remark that conditions (1), (2) of Theorem 6 are necessary for nefness of T^*X. Since for $H(\Lambda, n)$ (4) holds automatically, and since for $H(\Lambda, n)$, $\widehat{L}'_\alpha \cdot \widehat{L}'_\alpha = (1/n_\alpha)\deg(\pi|_{\widehat{L}'_\alpha})L'_\alpha \cdot L'_\alpha$, we have our equivalence.

Example 4.2. In [4] Sommese proves density of the Chern ratios $c_1^2(S)/c_2(S)$ in the interval $[1/5, 3]$ for the class of minimal compact complex surfaces S of general type. His method of exhibiting density in the interval $[2, 3]$ relies on Hirzebruch's surface $H(\Lambda, 5)$ where Λ is the $A_1(6)$ arrangement of lines in $\mathbf{P}^2$ (see [2, 4]). One has $c_1^2(H(\Lambda, 5)) = 3c_2(H(\Lambda, 5))$ and also that there is a fibering $f\colon H(\Lambda, 5) \to C$ where C is a Riemann surface of genus $g(C) = 6$. For any branched covering $F\colon C' \to C$, having branch locus in C that is disjoint from the image in C under f of the set where f is not of maximal rank, Sommese shows that the surfaces $S = F^*(H(\Lambda, 5)) = C' \times_C H(\Lambda, 5)$ have Chern ratios $c_1^2(S)/c_2(S)$ dense in the interval $[2, 3]$. $H(\Lambda, 5)$ has ample cotangent bundle [4], and $F^*(H(\Lambda, 5))$ branch covers $H(\Lambda, 5)$ with ramification set in $F^*(H(\Lambda, 5))$ consisting of the fibers B_j over the ramification set in C'. Each B_j has genus 76 [4], and clearly $B_j \cdot B_j = 0$. By Theorem 4 the surfaces $F^*(H(\Lambda, 5))$ have nef cotangent bundle. However, $T^*F^*(H(\Lambda, 5))$ is not ample as is easily seen. First note that $TF^*(H(\Lambda, 5))|_{B_j} \approx TB_j \oplus NB_j$. Let R be the curve in $\mathbf{P}(T^*F^*(H(\Lambda, 5))$ corresponding to B_j and NB_j. Then for $\xi = \xi_{T^*F^*(H(\Lambda, 5))}$ one has that $\xi^{-1}\cdot R = NB_j \cdot B_j = B_j \cdot B_j = 0$. Thus $\xi \cdot R = 0$, giving that $T^*F^*(H(\Lambda, 5))$ is not ample. This gives a class of surfaces X with T^*X nef but not ample, and $c_1^2(X)/c_2(X)$ is dense in the interval $[2, 3]$.

Example 4.3. The $F^*(H(\Lambda, 5))$ of Example 4.2 can also be seen as an example of Theorem 3. $F^*(H(\Lambda, 5))$ branch covers $H(\Lambda, 5)$ that in turn branch covers $\mathbf{BP}^2$. Since $F\colon C' \to C$ need not be galois, $F^*(H(\Lambda, 5))$ need not be galois over $\mathbf{BP}^2$. But Theorem 3 is applicable in this setting. One checks from the facts given for $H(\Lambda, 5)$ in Examples 4.2 that Theorem 3(1)–(3) hold and $T^*F^*(H(\Lambda, 5))$ is again nef.

Example 4.4. Let $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N$ be a general arrangement of pencils in $\mathbf{P}^2$; blowup all the points in the base loci to obtain $\mathbf{BP}^2$. Let $f_i\colon R_i \to \mathbf{P}^1$ be any Riemann surface that branch covers $\mathbf{P}^1$ having as branch locus for f_i the set $\pi_i(\Lambda_i') = \{\pi_i(C_{ij}') | C_{ij} \in \Lambda_i\}$, where $\pi_i : \mathbf{BP}^2 \to \mathbf{P}^1$ is the natural projection associated to the pencil P_i belonging to Λ_i, C_{ij}' is the proper transform of C_{ij}, and $\Lambda_i' = \{C_{ij}' | C_{ij} \in \Lambda_i\}$.

Construct X contained in $M := R_1 \times R_2 \times \cdots \times R_N \times \mathbf{BP}^2$ by requiring that $X = \{(x_1, x_2, \ldots, x_N, p) \in M | \pi_i(p) = f_i(x_i) \text{ for } i = 1, \ldots, N\}$. X is a smooth surface. This follows from the fact that the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & 0 & \cdots & 0 & -\frac{\partial \pi_1}{\partial x} & -\frac{\partial \pi_1}{\partial y} \\ 0 & \frac{\partial f_2}{\partial x_2} & & 0 & -\frac{\partial \pi_2}{\partial x} & -\frac{\partial \pi_2}{\partial y} \\ 0 & \cdots & \frac{\partial f_i}{\partial x_i} & 0 & -\frac{\partial \pi_i}{\partial x} & -\frac{\partial \pi_i}{\partial y} \\ 0 & \cdots & \cdots & \frac{\partial f_N}{\partial x_N} & -\frac{\partial \pi_N}{\partial x} & -\frac{\partial \pi_N}{\partial y} \end{bmatrix}$$

is of maximal rank. The left most N by N minor is of nonzero determinant at $(x_1, x_2, \ldots, x_N, p)$ unless $p \in C_{ij}'$ for some $C_{ij}' \in \Lambda'$ and $\partial f_i(x_i)/\partial x_i = 0$. In the case that the left most minor has determinant 0, we have that since Λ is a general pencil, at most two C_{ij}', say C_{ij}' and C_{IJ}', can intersect at p and they would meet transversely. Since C_{ij}' and C_{IJ}' are nonsingular fibers of π_i and π_I, respectively, one has that

$$\det\begin{pmatrix} \frac{\partial \pi_i}{\partial x} & \frac{\partial \pi_i}{\partial y} \\ \frac{\partial \pi_I}{\partial x} & \frac{\partial \pi_I}{\partial y} \end{pmatrix} \neq 0 \quad \text{at } p.$$

This compensates for the zero terms $\partial f_i(x_i)/\partial x_i$ and $\partial f_I(x_I)/\partial x_I$ and gives maximal rank N, so X is smooth.

Now X branch covers $\mathbf{BP}^2$, with $\pi : X \to \mathbf{BP}^2$ being the restriction to X of the projection of M onto the factor $\mathbf{BP}^2$, and π is ramified precisely over Λ'. This is seen by using the implicit function theorem in conjunction with the above $N \times (N+2)$ matrix. Since Λ is a general arrangement of pencils, let $\pi_1 \times \pi_2 \times \cdots \times \pi_k : \mathbf{BP}^2 \to \mathbf{P}^1 \times \mathbf{P}^1 \times \cdots \times \mathbf{P}^1$ be of maximal rank except at EC union finitely many points. If $g(R_i) \geq 1$ for $i = 1, \ldots, k$ then the Stein factorizations SC_i (of $\pi_i \circ \pi$) factor through R_i to $\mathbf{P}^1$ giving that $g(SC_i) \geq g(R_i) \geq 1$ for $i = 1, \ldots, k$. One then concludes by maximality of rank of $\pi_1 \times \pi_2 \times \cdots \times \pi_k$ that if $C_j \in \Lambda'$ then C_j must map onto say the ith factor $\mathbf{P}^1$ for some $i = 1, \ldots, k$. Hence each irreducible component $\widehat{C}_j$ of $\pi^{-1}(C_j)$ maps onto SC_i over the ith factor $\mathbf{P}^1$ and $e(\widehat{C}_j) \leq 0$. However, in order to guarantee that each $\widehat{C}_j \in \pi^{-1}(EC)$ has $e(\widehat{C}_j) \leq 0$, we must assume that $g(R_i) \geq 1$ for $i = 1, \ldots, k, k+1, \ldots, N$. Since each $\widehat{C}_j$ has nonpositive self-intersection, all the hypotheses of Theorem 5 hold. We conclude that if each genus $g(R_1)$, $g(R_2), \ldots, g(R_N)$ is positive then T^*X is nef.

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