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A DECOMPOSITION OF ELEMENTS OF THE FREE ALGEBRA

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ABSTRACT. Let f be an element of $F[X]$, the free associative algebra over a field F and n the maximum of the degrees of the variables and the multiplicities of the degrees in f. A partial ordering on the homogeneous elements of $F[X]$ is defined such that if f is homogeneous and $\operatorname{char} F \nmid n!$, then f can be decomposed into a sum of two polynomials f_0 and f_1 such that for $0 < m \leq n$, f_0 is symmetric or skew symmetric in all its arguments of degree m depending on whether m is even or odd and f_1 is a consequence of polynomials of lower type than f. Osborn's Theorem about the symmetry of the absolutely irreducible polynomial identities is obtained as a corollary. The same holds in the free nonassociative algebra. The proofs are combinatorial.

0. INTRODUCTION

In [2] Osborn introduced a partial ordering on the free nonassociative algebra $F\langle X\rangle$ over a field F and proved the following interesting result: if n is a positive integer with $\operatorname{char} F \nmid n!$ and f is an identity of a not necessarily associative algebra A over F such that A has no identity of type lower than f in the partial ordering, then f is symmetric or skew symmetric in its arguments of degree n depending on whether n is even or odd. This theorem was used in [1] to determine the identities of degree $2n$ of the space of $n \times n$ symmetric matrices; however, the assumption that A has no identity of lower type limits the use of the result in many cases. In this paper we remove the restriction that A has no identities of lower type and consider polynomials at large, not only polynomial identities. Osborn's Theorem follows as a consequence.

Throughout the paper F always denotes a field. The polynomials are in the free associative algebra $F[X]$, where X is a set of countable noncommuting indeterminates $x_1, x_2, \ldots$ over the field F. All identities mentioned are homogeneous *weak identities* (i.e., polynomials that evaluate to zero on some fixed subspace V of an algebra A) except when otherwise noted.

1. A PARTIAL ORDER, Δ-OPERATORS, AND WEAK T-IDEALS

Following Osborn [2, p. 78] we introduce a partial ordering on the set of homogeneous polynomials in $F[X]$, the free associative algebra in the noncommuting variables $X = \{x_1, x_2, \ldots\}$. If $p(x_1, \ldots, x_m)$ is a homogeneous

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polynomial of degree n, as in [3], we say that p is of *type* $[n_1, \dots, n_m]$ if n_j is the degree of x_j in p and $n_m \neq 0$ but $n_j = 0$ for $j > m$. Define $[[n_1, \dots, n_m]]$ to be $[n_{i_1}, \dots, n_{i_m}]$, where $n_{i_1} \geq n_{i_2} \geq \cdots \geq n_{i_m}$ and $\{n_{i_1}, \dots, n_{i_m}\} = \{n_1, \dots, n_m\}$ as multisets. Let p' be a homogeneous polynomial of degree n' and of type $[n'_1, \dots, n'_{m'}]$. If

$$[[n'_1, n'_2, \dots, n'_{m'}]] = [n'_{j_1}, n'_{j_2}, \dots, n'_{j_{m'}}]$$

then p is lower than p' in the *partial ordering* if and only if either (i) $n < n'$ or (ii) $n = n'$ and $n_{i_k} > n'_{j_k}$ for the first integer k such that $n_{i_k} \neq n'_{j_k}$; otherwise, the two polynomials are not comparable. If an integer n_j is repeated k times, we shall denote this by an exponent; for example, $[3, 2^2, 1^3]$ means $[3, 2, 2, 1, 1, 1]$.

Let $f(x_1, x_2, \dots, x_i, \dots, x_j, \dots, x_m) \in F[X]$ be a homogeneous polynomial and x_i, x_j have degree n_i, n_j in f, respectively. Then the polynomial

$$\begin{aligned} &f'(x_1, \dots, x_i, \dots, x_j, \dots, x_m) \\ &\quad := f(x_1, \dots, x_i + x_j, \dots, x_j, \dots, x_m) \\ &\quad = f(x_1, \dots, x_i, \dots, x_j, \dots, x_m) + f_1 + \cdots + f_{n_i}, \end{aligned} \tag{1}$$

where f_k is the homogeneous component of f' and x_i, x_j have degree $n_i - k$, $n_j + k$ in f_k, respectively. Each f_k is obtained by setting k x_i's to be x_j in f. Following [3, p. 9] we define

$$\Delta^k(x_i, x_j)f := f_k, \quad k = 1, 2, \dots, n_i, \qquad \Delta^0(x_i, x_j)f := f.$$

The mapping Δ^k is called a Δ-*operator* (or a *derivation*) and $\Delta^k(x_i, x_j)f$ is called a *partial linearization* of f. If $n_i > 0$ and $n_j = 0$ (which is allowed in the above definition) then $\Delta^1(x_i, x_j)f$ is of degree $n_i - 1$ in x_i and 1 in x_j. Therefore, if enough variables of degree 0 were present then one could linearize f using successive Δ-operators. We denote by Δf the set of all polynomials in $F[X]$ that can be obtained from f by means of repeated Δ-operations

$$\Delta f = \{g \in F[X] \mid g = \Delta^{j_1}(x_{i_1}, x_{k_1}) \cdots \Delta^{j_s}(x_{i_s}, x_{k_s})f\}.$$

It is well known that the set $T(A)$ of all polynomial identities of an F-algebra is a T-ideal, i.e., an ideal of $F[X]$ that is invariant under all endomorphisms of $F[X]$. However, this is no longer true for weak identities. Hence we introduce the concept of weak T-ideals.

Definition 1. An ideal W of $F[X]$ is called a *weak T-ideal* if W is invariant under every linear mapping $x_i \mapsto \sum \alpha_{ij} x_j$ for $\alpha_{ij} \in F$.

If S is a subset of $F[X]$ then the smallest weak T-ideal containing S in $F[X]$ is called the weak T-ideal generated by S and denoted by $\langle S \rangle$.

One basic example of weak T-ideal is the ideal $T(A, V)$ consisting of all weak polynomial identities on a subspace V of an algebra A.

Using a standard Vandermonde argument (e.g., [3, Theorem 5, p. 12]) one can prove

Proposition 1. *Let $f(\dots, x_i, \dots, x_j, \dots) \in F[X]$ be a homogeneous polynomial of type $[\dots, n_i, \dots, n_j, \dots]$. If $|F| > n_i$ then $\Delta^k(x_i, x_j)f \in \langle f \rangle$,*

$k = 0, 1, \ldots, n_i$. *In particular, if* $f \in T(A, V)$ *then* $\Delta^k(x_i, x_j)f \in T(A, V)$ *for* $k = 0, 1, \ldots, n_i$.

Let $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ be homogeneous polynomials of $F[X]$. We shall say that g is a *consequence of* (or *comes from*) the polynomial f if g belongs to the vector space $\operatorname{Span}\{\Delta f\}$.

Since a partial linearization of an identity f is not necessarily an identity, the polynomials in Δf need not be identities either. But from Proposition 1, if F contains enough elements then Δf lies in the T-ideal $\langle f \rangle$, and hence if g comes from f then g is a linear combination of identities in $\langle f \rangle$.

2. The main results

As in [3], the free associative algebra $F[X]$ has a decomposition into a direct sum of subspaces $V^{[n_1, \ldots, n_k]}[X]$ consisting of the homogeneous polynomials of type $[n_1, \ldots, n_k]$. In this section we give a decomposition of the elements of $V^{[n_1, \ldots, n_k]}[X]$, which is useful in the study of identities.

Theorem 1. *Let* n *be a positive integer and let* p *be a homogeneous polynomial with coefficients in a field* F *of characteristic not dividing* $n!$. *Let* x *and* y *be arguments of degree* n. *Define* p_1 *by*

$$p_1 := p(\ldots, y, \ldots, x, \ldots) - (-1)^n p(\ldots, x, \ldots, y, \ldots).$$

Then the following statements hold:

(1) *The polynomial* p_1 *comes from a polynomial of lower type than* p.

(2) *Let* $\operatorname{char} F \neq 2$. *If there exist two variables in* p, *either of even degree in which* p *is skew symmetric or of odd degree in which* p *is symmetric, then* $p = \frac{1}{2}p_1$ *comes from a polynomial of lower type.*

(3) *If* $|F| > 2n - 1$ *and* p *is an identity, then* p_1 *is a linear combination of identities of type lower than* p.

(4) *If* $\Delta(y, x)p = 0$ *then* p *is either symmetric or skew symmetric in* x *and* y *depending on whether* n *is even or odd.*

Proof. Define

$$\tilde{p}_1 := \alpha_0 f_n + \alpha_1 f_{n-1} + \cdots + \alpha_{n-1} f_1,$$

where $\alpha_k = (-1)^k k!/n(n-1)\cdots(n-k)$, $0 \leq k \leq n-1$,

$$\begin{aligned}
f_1 &:= \Delta^0(y, x)\Delta^1(x, y)\Delta^1(y, x)p, \\
f_2 &:= \Delta^1(y, x)\Delta^2(x, y)\Delta^2(y, x)p - \binom{n}{1} f_1, \\
f_{m+1} &:= \Delta^m(y, x)\Delta^{m+1}(x, y)\Delta^2(x, y)p \\
&\quad - \binom{n}{m} f_1 - \binom{n-1}{m-1} f_2 - \cdots - \binom{n-m+1}{1} f_m.
\end{aligned} \tag{2}$$

The polynomial $\tilde{p}_1$ comes from $\Delta^1(y, x)p$, which is of lower type than p.

If $|F| > 2n - 1$ and p is an identity, then for each i, $i = 1, \ldots, n$, $f_i \in \langle \Delta^1(y, x)p \rangle$ by Proposition 1 and hence $\tilde{p}_1$ is a linear combination of consequences of lower type identities f_i. Thus, to prove statements (1) and (3) of the theorem, it suffices to show that

$$(3) \qquad p(\dots, y, \dots, x, \dots) = (-1)^n p(\dots, x, \dots, y, \dots) + \tilde{p}_1.$$

Since $\Delta^k(x, y)$ and $\Delta^m(y, x)$ change only the variables x and y, the polynomials $p(\dots, y, \dots, x, \dots)$, $(-1)^n p(\dots, x, \dots, y, \dots)$, and $\tilde{p}_1$ are of the same type. So, they are linear combinations of the same monomials. Thus, to show (3) it suffices to show that for each monomial M, the coefficient of M in $p(\dots, y, \dots, x, \dots)$ and the coefficient of M in $(-1)^n p(\dots, x, \dots, y, \dots)+$ $\tilde{p}_1$ are the same.

First we define an equivalence relation on the set of monomials occurring in $p : M_1 \sim M_2$ if M_2 is obtainable from M_1 by permuting some x's and y's and leaving the other variables fixed. We denote by $[M]$ the equivalence class of M. Let I be the set of all associative words in x and y involving exactly n x's and n y's. Setting the variables, other than x and y, equal to 1 in an element of $[M]$ yields a bijection from $[M]$ to I; so we may write $[M] = \{M_i \mid i \in I\}$ and, in particular, $M = M_w$ for some $w \in I$. Let $I^{(k)}$, $k = -1, 0, 1, \dots, n$, be the set of all associative words that have $n+k$ y's and $n-k$ x's. Then $I = I^{(0)}$ and $I' := I^{(-1)}$ is the set of all associative words in x and y involving exactly $n+1$ x's and $n-1$ y's. For $i, j \in I' \cup I \cup I^{(1)} \cup \dots \cup I^{(n)}$, we define $i \cdot j$ to be the number of positions in which both i and j have a y.

Let c_i be the coefficient of M_i in p and r_j be the coefficient of M_j in $p' := \Delta^1(y, x)p$. Then for each $j \in I'$ the total coefficient of M_j in p' will be the sum of the $n+1$ coefficients c_i where i runs over all those elements of I such that $i \cdot j = n-1$, i.e.,

$$(4) \qquad r_j = \sum_{i \cdot j = n-1} c_i \quad \forall j \in I'.$$

If $k = j \cdot w$ then every i occurring in (4) satisfies either $i \cdot w = k$ or $i \cdot w = k+1$. Indeed since j has $n-1$ elements y's and $i \cdot j = n-1$, i has a y in a given position whenever j does. So in the positions where both j and w have a y, i has a y also. Therefore $j \cdot w = k$ implies $i \cdot w$ is at least k. But i has only one more y than j, so $i \cdot w \le k+1$. Each $i \in I$ satisfying $i \cdot w = k$ occurs in exactly $n-k$ equations of type (4) for which $j \cdot w = k$. Indeed i occurs in (4) iff M_i becomes M_j when replacing one y by an x in M_i; since $i \cdot w = k$, if we put one y in M_i that occurs in the same position in w to be x to get M_j, then $j \cdot w = k-1$ not k. This cannot happen, so we have only $n-k$ choices and each one produces distinct M_j's, so c_i occurs $n-k$ times in the set of equations of type (4) satisfying $j \cdot w = k$. Each $i \in I$ satisfying $i \cdot w = k+1$ occurs in exactly $k+1$ equations of type (4) satisfying $j \cdot w = k$, for, this time, we can only choose those y's that occur in the same position in i and w in order to get $j \cdot w = k$.

Now we add up all equations of type (4) satisfying $j \cdot w = k$ to get

$$(5) \qquad \sum_{j \cdot w = k} r_j = (n-k)C_k + (k+1)C_{k+1}, \qquad k = 0, 1, 2, \dots, n-1,$$

where, for $0 \le m \le n$, $C_m := \sum_{i \cdot w = m} c_i$, with c_i occurring in an equation of type (4) satisfying $j \cdot w = k$. Substituting the n equations of (5) into one another yields

$$C_0 = (-1)^n C_n + (-1)^{n-1}\frac{(n-1)!}{n!}\sum_{j\cdot w=n-1} r_j + (-1)^{n-2}\frac{(n-2)!}{n(n-1)\cdots 2}$$
$$\times \sum_{j\cdot w=n-2} r_j + \cdots - \frac{1}{n(n-1)}\sum_{j\cdot w=1} r_j + \frac{1}{n}\sum_{j\cdot w=0} r_j.$$

We know that $C_n = c_w$ and $C_0 = c_{w'}$, where w' is the word obtained from w by interchanging all the x's and y's. Thus we have that

$$c_{w'} = (-1)^n c_w + \alpha_{n-1}\beta_{n-1} + \cdots + \alpha_0\beta_0,$$

where $\alpha_k = (-1)^k k!/n\cdots(n-k)$ and $\beta_k = \sum_{j\cdot w=k} r_j$, $k = 0, 1, \dots, n-1$. Next we show that $\sum_{i=0}^{n-1}\alpha_i\beta_i$ is the coefficient of the monomial M_w in $\tilde{p}_1$ and, therefore, $\tilde{p}_1$ comes from p'. By the definition of $\tilde{p}_1$, it suffices to show that each β_{n-k} is the coefficient of M_w in f_k. This will be proved by induction on k.

For $v, u, u' \in I \cup I' \cup I^{(1)} \cup \cdots \cup I^{(n)}$ we define $\#(v) :=$ the degree of y in v, and we say that $u \le u'$ if whenever u has a y in a given position so does u'. Define $u \cup u'$ to be the word that has a y in a given position iff u or u' does (e.g., if $u = xxyxyy$ and $u' = yxyxxx$, then $u \cup u' = yxyxyy$). From the definition we have that if $u \le t$ and $u' \le t$ then $u \cup u' \le t$ and $\#(u\cup u') \le \#(t)$.

Next let us consider β_{n-1}. Since f_1 is obtained by setting one x equal to y in p', the coefficient of M_w in f_1 is

$$\gamma_w = \sum_{j\cdot w=n-1} r_j = \beta_{n-1}.$$

Suppose $\beta_{n-1}, \beta_{n-2}, \dots, \beta_{n-m}$ are the coefficients of M_w in $f_1, f_2, \dots, f_m$, respectively. Now for β_{n-m-1}, putting $m+1$ x's to be y in $\Delta^1(y,x)p$ yields the polynomial $\Delta^{m+1}(x,y)\Delta^1(y,x)p$, then putting m elements y's to be x in $\Delta^{m+1}(x,y)\Delta^1(y,x)p$ we get the polynomial $\Delta^m(y,x)\Delta^{m+1}(x,y)\Delta^1(y,x)p$. In $\Delta^{m+1}(x,y)\Delta^1(y,x)p$, the monomial M_t has coefficient $\lambda_t = \sum_{j\cdot t=n-1} r_j$, since $\#(j) = n-1$ and $\#(t) = n+m = (n-1)+(m+1)$. (If $j\cdot t < n-1$ then putting $m+1$ x's to be y in M_j cannot yield M_t, which has $n+m$ y's). Similarly, in $\Delta^m(y,x)\Delta^{m+1}(x,y)\Delta^1(y,x)p$, M_w has coefficient $\mu_w = \sum_{t\in I^{(m)},\ t\cdot w=n}\lambda_t$, since $\#(t) = n+m$ and $\#(w) = n$. Thus

$$\mu_w = \sum_{t\in I^{(m)},\ t\cdot w=n}\left(\sum_{j\in I',\ j\cdot t=n-1} r_j\right). \tag{6}$$

We claim that

$$(*)\qquad \begin{aligned}&\{j\in I' \mid j\cdot t = n-1 \text{ for some } t\in I^{(m)} \text{ and } t\cdot w = n\}\\ &\quad = \{j\in I' \mid n-m-1 \le j\cdot w \le n-1\}.\end{aligned}$$

For if $j\in I'$ with $j\cdot t = n-1$, $\#(t) = n+m$, and $t\cdot w = n$, then $j \le t$, $w \le t$, and $j\cup w \le t$. Since $\#(j) = n-1$, $j\cdot w \le n-1$. If $j\cdot w \le n-m-2$ then $j\cup w \le t$ implies that $\#(j\cup w) \le n+m$. But

$$\#(j\cup w) = n-1+(n-j\cdot w) \ge n-1+n-n+m+2 = n+m+1,$$

a contradiction. On the other hand, if $j \cdot w = n-k$, $k = 1, 2, \ldots, m+1$, then

$$\#(j \cup w) = n-1+n-(n-k) = n-1+k \leq n+m;$$

therefore, there exists a word $t \in I^{(m)}$ such that $j \cup w \leq t$. Since $j \leq t$ and $w \leq t$, $j \cdot t = \#(j) = n-1$ and $w \cdot t = \#(w) = n$. This establishes $(*)$. It also means that r_j occurs in (6) iff $j \cdot w = n-k$ for some k with $1 \leq k \leq n-m-1$.

Each r_j may occur several times in (6). We group the r_j's in (6) depending on $j \cdot w = n-k$ for each k. We claim that

(**) each r_j with $j \in I'$ satisfying $j \cdot w = n-k$ occurs $\binom{n-k+1}{m-k+1}$ times in (6).

Indeed, the number of the occurrences of r_j in (6) is exactly the number of words $t \in I^{(m)}$ such that $t \cdot w = n$ and $j \cdot t = n-1$; that is, $t \in I^{(m)}$ such that $w \cup j \leq t$ since $\#(j) = n-1$. Now if $w \cdot j = n-k$ then

$$\#(w \cup j) = n+n-1-n+k = n+k-1.$$

So the word t obtained by changing $m-k+1$ x's in $w \cup j$ to be y's belongs to $I^{(m)}$ and $w \cup j \leq t$; however, the number of choices for such a word t is $2n-(n+k-1)$ choose $m-k+1$, that is, $\binom{n-k+1}{m-k+1}$. Hence $(**)$ holds and

$$\mu_w = \sum_{t \cdot w = n} \sum_{j \cdot t = n-1} r_j = \binom{n}{m} \sum_{j \cdot w = n-1} r_j + \binom{n-1}{m-1} \sum_{j \cdot w = n-2} r_j$$
$$+ \cdots + \binom{n-m+1}{1} \sum_{j \cdot w = n-m} r_j + \sum_{j \cdot w = n-m-1} r_j.$$

Since $\beta_{n-k} = \sum_{j \cdot w = n-k} r_j$,

$$\beta_{n-m-1} = \mu_w - \binom{n}{m} \beta_{n-1} - \binom{n-1}{m-1} \beta_{n-2} - \cdots - \binom{n-m-1}{1} \beta_{n-m}.$$

Thus, by the induction hypothesis and the definition of f_{m+1}, β_{n-m-1} is the coefficient of M_w in f_{m+1}. Therefore, $\sum \alpha_k \beta_k$ is the coefficient of M_w in $\tilde{p}_1$, from the definition of $\tilde{p}_1$.

We also know that $c_{w'}$ and $(-1)^n c_w$ are the coefficients of M_w in $p(\ldots, y, \ldots, x, \ldots)$ and $(-1)^n p(\ldots, x, \ldots, y, \ldots)$, respectively. Thus (3) holds. So statements (1) and (3) are proved. The second statement of the theorem follows easily from (3).

If $\Delta(y, x)p = 0$ then $\tilde{p}_1 = 0$ by the definition of $\tilde{p}_1$. Hence $p_1 = 0$, which implies that p is symmetric or skew symmetric in x and y depending on whether n is even or odd. □

Since the associativity of $F[X]$ was not used in the proof of Theorem 1, Theorem 1 is also true for the free nonassociative algebra. The proof is exactly the same except for the definition of the mapping from $[M]$ to I. In this case, $[M]$ consists of monomials having exactly the same distribution of parentheses and that differ only in the positions of x and y. When all variables other than x and y are set equal to 1, a monomial in x, y is obtained. Since the parentheses distribution is always the same, we may therefore drop the parentheses and thus obtain an element of I. So again there is a bijection from $[M]$ to I. Thus we have

Theorem 1′. *Let n be a positive integer. Let p be a homogeneous polynomial $F\langle X\rangle$, the free nonassociative algebra in the variables $X = \{x_1, x_2, \dots\}$ over a field F of characteristic not dividing $n!$. Let x and y be arguments of degree n. Define p_1 by*

$$p_1 := p(\dots, y, \dots, x, \dots) - (-1)^n p(\dots, x, \dots, y, \dots).$$

Then the following statements hold:

(1) *The polynomial p_1 comes from a polynomial of lower type than p.*

(2) *Let* $\operatorname{char} F \neq 2$. *If there exist two variables in p of even degree in which p is skew symmetric or of odd degree in which p is symmetric, then $p = \frac{1}{2}p_1$ comes from a polynomial of lower type.*

(3) *If $|F| > 2n-1$ and p is an identity, then p_1 is a linear combination of identities of type lower than p.*

(4) *If $\Delta(y, x)p = 0$ then p is symmetric or skew symmetric in x and y depending on whether n is even or odd.*

From Theorem $1'$ we have

Corollary 1 [Osborn's Theorem]. *Let p be a homogeneous identity in a nonassociative algebra A over a field F of characteristic not dividing $n!$. If A has no identity of type lower than p then p is symmetric or skew symmetric in its arguments of degree n, depending on whether n is even or odd.*

Theorem 1 holds for varieties of nonassociative algebras. Given an element p in the free algebra of the variety, consider p as an element of the free nonassociative algebra. Equation (3) holds in the free nonassociative algebra and, passing to the quotient algebra, it holds also in the free algebra of the variety.

Theorem 1 says that for each pair or arguments x, y of degree n there exists a polynomial $p' = \frac{1}{2}(p(\dots, x, \dots, y, \dots) - (-1)^n p(\dots, y, \dots, x, \dots))$ that comes from polynomials of lower type such that $p - p'$ is symmetric or skew symmetric in x, y depending on whether n is even or odd. In fact, we can find a polynomial p_1 that comes from some polynomials of type lower than that of p such that $p - p_1$ is symmetric or skew symmetric in all its arguments of degree n, depending on whether n is even or odd.

Theorem 2. *Let r be an integer and F be a field with* $\operatorname{char} F \nmid r!$. *Let V be an $\mathcal{S}_r$-module. Then*

1. $V = \{v \in V \mid \pi v = v, \forall \pi \in \mathcal{S}_r\} + (\sum_{1\le i<j\le r}\{v \in V \mid (i, j)v = -v\})$;
2. $V = \{v \in V \mid \pi v = -v, \forall \pi \in \mathcal{S}_r\} + (\sum_{1\le i<j\le r}\{v \in V \mid (i, j)v = v\})$.

Proof. To prove the theorem we use induction on r. In what follows, we agree that the product of two permutations is performed from right to left. For $r = 1$, the theorem holds trivially. For $r = 2$, every $v \in V$ has a unique decomposition

$$v = \tfrac{1}{2}(v + (12)v) + \tfrac{1}{2}(v - (12)v).$$

Suppose the result is true for $r-1$. We imbed $\mathcal{S}_{r-1}$ into $\mathcal{S}_r$ as $\{\pi \in S_r \mid \pi(r) = r\}$. Thus a given $\mathcal{S}_r$-module is also an $\mathcal{S}_{r-1}$-module. Hence by induction,

$$V = \{v \in V \mid \pi v = v, \forall \pi \in \mathcal{S}_{r-1}\} + \left(\sum_{1\le i<j\le r-1}\{v \in V \mid (i, j)v = -v\}\right).$$

Let $v \in V^{\mathscr{S}_{r-1}} := \{v \in V \mid \pi v = v\,, \forall \pi \in \mathscr{S}_{r-1}\}$ and define

$$w := v - \frac{1}{r} \sum_{1 \le i \le r-1} (v - (ir)v).$$

It suffices to show $w \in V^{\mathscr{S}_r}$. We have

$$w = \frac{1}{r}\left(v + \sum_{1 \le i \le r-1} (ir)v\right),$$

hence for $1 \le j \le r-1$,

$$(jr)w = \frac{1}{r}\left[(jr)v + v + \sum_{1 \le i \le r-1,\ i \ne j} (jr)(ir)v\right] = w,$$

since $(jr)(ir) = (ir)(ij)$, and thus

$$(jr)(ir)v = (ir)(ij)v = (ir)v,$$

using $v \in V^{\mathscr{S}_{r-1}}$. Because $\{(jr) \mid 1 \le j \le r-1\}$ generates $\mathscr{S}_r$, we have shown that $w \in V^{\mathscr{S}_r}$, completing the proof of Theorem 2(1).

Symmetrically, to prove Theorem 2(2) we define

$$w' := v' - \frac{1}{r} \sum_{1 \le i \le r-1} (v' + (ir)v'),$$

where $v' \in \{v \in V \mid \pi v = -v\,, \forall \pi \in \mathscr{S}_{r-1}\}$. Now we can repeat the proof of (1) of Theorem 2 step-by-step to show that $w' \in \{v \in V \mid \pi v = -v\,, \forall \pi \in \mathscr{S}_r\}$. This completes the proof of Theorem 2. □

Let $V = V^{[m^r, n^s, \dots, u^t]}[X]$, and

$$W_1 = \mathrm{Span}\{f(x_1, \dots, x_r, y_1, \dots, y_s, \dots) \in V \mid f \text{ is skew symmetric in } x_i \text{ and } x_j \text{ for some } i \ne j\}.$$

Let U be the set of the element $f(x_1, \dots, x_r, y_1, \dots, y_s, \dots, z_1, \dots, z_t) \in V$ that are symmetric or skew symmetric in some w_i, w_j where $i \ne j$ and $w \in \{x, y, \dots, z\}$, depending on whether the degree of w_i in f is odd or even. Let $V_1 = \mathrm{Span}\, U$. Then we have

Theorem 3. *Let $m_0 = \max\{m, n, \dots, u, r, s, \dots, t\}$. If $\mathrm{char}\, F \nmid m_0!$ then every element f in V has a decomposition into a sum of two polynomials of the same type as f, $f = f_0 + f_1$, where for each k with $0 < k \le m_0$, f_0 is symmetric or skew symmetric in all variables of degree k depending on whether k is even or odd and $f_1 \in V_1$. Moreover V_1 is spanned by elements that come from lower type polynomials: if $g \in V_1$ then $g = \sum g_k$, where $g_k \in U$ and g_k is given by formula* (2).

Proof. Let $f(x_1, \dots, x_r, y_1, \dots, y_s, \dots, z_1, \dots, z_t)$ in V. First we show that $f = f_0 + f_1$, where $f_1 \in V_1$ and f_0 is symmetric or skew symmetric in all x_i's depending on whether m is even or odd. For an arbitrary $\sigma \in \mathscr{S}_r$ we define

$$\begin{aligned}&\sigma(f(x_1, \dots, x_r, y_1, \dots, y_s, \dots, z_1, \dots, z_t))\\ &\qquad := f(x_{\sigma(1)}, \dots, x_{\sigma(r)}, y_1, \dots, y_s, \dots, z_1, \dots, z_t).\end{aligned}$$

Then V is an $\mathcal{S}_r$-module. If m is even then by Theorem 2(1), $f = f_0 + f_1$ where $f_1 \in W_1$ and f_0 is symmetric in all x_i's. Since $W_1 \subseteq V_1$ by the definitions of W_1 and V_1, we get $f_1 \in V_1$. If m is odd then we use Theorem 2(2).

In any case, we have that $f = h_0 + h_1$, where $h_1 \in V_1$ and h_0 is symmetric or skew symmetric in all x_i's depending on whether m is even or odd. Next we consider h_0 and variables $y_1, \dots, y_s$ as above. The above process will not destroy the symmetry of h_0 in the x_i's. Thus after a finite number of steps, we have that $f = f_0 + f_1$, where $f_1 \in V_1$ and f_0 is as in the theorem.

The last statement of the theorem follows from Theorem 1 and the definition of U. □

To summarize, if $f \in T(A, W) \cap V$ then $f = f_0 + f_1$ with f_i an identity, for $i = 0, 1$ by Theorem 3 and the fact that $T(A, W) \cap V$ is an $\mathcal{S}_r$-module. But we do not know whether f_1 comes from the identity of lower type although it comes from a lower type polynomial. However, using Proposition 1 we have

Corollary 2. *Let $m_0 = \max\{m, n, \dots, u, r, s, \dots, t\}$. If $\operatorname{char} F \nmid m_0!$ and $|F| > 2m_0 - 1$, then every identity f in V has a decomposition into a sum of two identities of the same type as f, $f = f_0 + f_1$, where for each k with $0 < k \leq m_0$, f_0 is symmetric or skew symmetric in all variables of degree k depending on whether k is even or odd and f_1 comes from lower type identities.*

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