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BOUNDARY VALUES OF HOLOMORPHIC SEMIGROUPS

KHRISTO BOYADZHIEV AND RALPH DELAUBENFELS

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ABSTRACT. Suppose A generates a bounded strongly continuous holomorphic semigroup of angle $\pi/2$. We show that iA generates a $(1-A)^{-r}$-regularized group, which is $O(1+|s|^r)$ $\forall r > \gamma \geq 0$, if and only if $\|e^{zA}\|$ is $O(((1+|z|)/\mathrm{Re}(z))^r)$ $\forall r > \gamma$ and iA generates a bounded $(1-A)^{-r}$-regularized group $\forall r > \gamma \geq 0$ if and only if $\|e^{zA}\|$ is $O((1/\mathrm{Re}(z))^r)$ $\forall r > \gamma$. We apply this to the Schrödinger operator $i(\Delta - V)$.

I. INTRODUCTION

The heat semigroup $e^{z\Delta}$, where Δ is the Laplacian on $L^p(\mathbf{R}^n)$, constitutes a holomorphic function on the right half-plane (RHP) that is bounded on every sector $S_\theta \equiv \{z \mid |\arg(z)| < \theta\}$ for θ less than $\pi/2$. Its boundary values $e^{is\Delta}$, however, constitute a strongly continuous group (the Schrödinger group) only when $p = 2$. It is important to "regularize" $e^{is\Delta}$ for $p \neq 2$ and study the analogous Schrödinger "group" in this case.

In general, when e^{zA} is a bounded (in sectors S_θ for $\theta < \pi/2$) holomorphic strongly continuous semigroup of angle $\pi/2$, it is natural to ask when boundary values exist, in some sense. As indicated in the first paragraph, to include many interesting examples, this "sense" needs to be weaker than the sense of a strongly continuous group.

When C is a bounded injective operator, a *C-regularized group* is a strongly continuous family of bounded operators $\{W(t)\}_{t\in\mathbf{R}}$ such that $W(0) = C$, $W(t)W(s) = CW(t+s)$ $\forall s, t \in \mathbf{R}$. The *generator* is defined by $Ax = C^{-1}(d/dt)W(t)x|_{t=0}$, with maximal domain. When A generates a C-regularized group, the (reversible) *abstract Cauchy problem*

$$\frac{d}{dt}u(t, x) = A(u(t, x)) \quad (t \in \mathbf{R}), \qquad u(0, x) = x$$

has a unique mild solution $u(t, x) = C^{-1}W(t)x$ for all initial data x in the image of C. We also obtain well-posedness on a subspace, that is, there exists a Fréchet space Z such that

$$[\mathrm{Im}(C)] \hookrightarrow Z \hookrightarrow X$$

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and A restricted to Z generates a strongly continuous group. When $W(t)$ is exponentially bounded, Z may be chosen to be a Banach space.

The following result has been proven in different mathematical languages in at least three different places. We state it here essentially in the language of regularized semigroups (see Theorem 3.1). This interpretation is more convenient because it gives us classical solutions.

Proposition 1.1. *When $r > n|\frac{1}{p} - \frac{1}{2}|$, $\{(1-\Delta)^{-r}e^{is\Delta}\}_{s\in\mathbf{R}}$ is a strongly continuous family of bounded operators on $L^p(\mathbf{R}^n)$.*

This appeared in [15] in the language of Fourier multipliers, in [1] in the language of smooth distribution groups, and in [9] in the language of integrated semigroups. In [2] and [11] results similar to Proposition 1.1 for $H = \Delta - V$ for appropriate potentials V appear.

In this paper, we show that when A generates a bounded strongly continuous holomorphic semigroup of angle $\pi/2$, the amount of regularizing, C, required to make e^{isA} into a C-regularized group $e^{isA}C$ depends on how rapidly $\|e^{zA}\|$ grows as z approaches the imaginary axis. This is useful for applications, because we may restrict our attention to *bounded* operators e^{zA} for $\mathrm{Re}(z)$ strictly greater than zero, which are much easier to work with than the unbounded operators e^{isA} whose definition, in general, may be somewhat mysterious. Proposition 1.1, along with a growth estimate on the regularized group, follows easily from our general result, since the heat semigroup is an integral operator, with a managable kernel, so that we may readily estimate the norm of $e^{z\Delta}$ (§III).

More specifically, we may look at the behaviour of $\|e^{zA}\|$ in sectors S_θ as θ approaches $\pi/2$ or in half-planes $\mathrm{Re}(z) > a > 0$ as a approaches 0. By definition of a bounded holomorphic semigroup, $\|e^{zA}\|$ is bounded in S_θ for any $\theta < \pi/2$; we show that iA generating a $(1-A)^{-r}$-regularized group that is $O(1+|s|^r)$ corresponds to $\|e^{zA}\|$ being $O((1/\cos\theta)^r)$, where $\theta = \arg(z)$. Generating a bounded $(1-A)^{-r}$-regularized group corresponds to $\|e^{zA}\|$ being $O((1/\mathrm{Re}(z))^r)$.

We also apply our general result to the Schrödinger operator with potential, $i(\Delta - V)$, for V real valued, with V_+ a Kato perturbation, and $V_- \in L^\infty(\mathbf{R}^n)$, and show that

$$\{(\omega - (\Delta - V))^{-r}e^{is(\Delta - V)}\}_{s\in\mathbf{R}}$$

is a strongly continuous polynomially bounded family of bounded operators on $L^p(\mathbf{R}^n)$ for $r > 2n|\frac{1}{p} - \frac{1}{2}|$, ω sufficiently large (Theorem 3.4). This improves some known results (see [2, 11]) with a shorter proof.

Basic material on strongly continuous semigroups may be found in [8, 12, 13] and on regularized semigroups in [3, 5, 6].

II. Main results

We assume throughout this section that A generates a bounded strongly continuous holomorphic semigroup $\{e^{zA}\}_{z\in\mathrm{RHP}}$ of angle $(\pi/2)$.

Theorem 2.1. *Suppose that $\gamma \geq 0$ and $\exists M < \infty$ such that*

$$\|e^{zA}\| \leq M\left(\frac{|z|}{\mathrm{Re}(z)}\right)^{\gamma}.$$

Then $\forall r > \gamma$ $\exists M_{r,\gamma} < \infty$ *such that* iA *generates a* $(1-A)^{-r}$*-regularized group* $\{W_r(s)\}_{s\in\mathbf{R}}$ *such that*

$$\|W_r(s)\| \le M_{r,\gamma}(1+|s|^\gamma).$$

Theorem 2.2. *Suppose* $\gamma \ge 0$. *Then the following are equivalent.*

(a) $\forall r > \gamma$ $\exists M_r < \infty$ *such that*

$$\|e^{zA}\| \le M_r \left(\frac{1+|z|}{\operatorname{Re}(z)}\right)^r.$$

(b) $\forall r > \gamma$ $\exists C_r$ *such that* iA *generates a* $(1-A)^{-r}$*-regularized group* $\{W_r(s)\}_{s\in\mathbf{R}}$ *such that*

$$\|W_r(s)\| \le C_r(1+|s|^r).$$

(c) $\|(1+|z|^r)^{-1}e^{zA}(1-A)^{-r}\|$ *is uniformly bounded in the right half-plane* $\forall r > \gamma$.

It is theoretically interesting that a *bounded* $(1-A)^{-r}$-regularized semigroup corresponds to the same growth conditions on right half-planes $\operatorname{Re}(z) > a$ rather than sectors. The proof of the following theorem is essentially the same as the proof of Theorem 2.2.

Theorem 2.3. *Suppose* $\gamma \ge 0$. *Then the following are equivalent.*

(a) $\forall r > \gamma$ $\exists M_r < \infty$ *such that* $\|e^{zA}\| \le M_r(\operatorname{Re}(z))^{-r}$.

(b) iA *generates a bounded* $(1-A)^{-r}$*-regularized group* $\forall r > \gamma$.

(c) $\|e^{zA}(1-A)^{-r}\|$ *is uniformly bounded in the right half-plane* $\forall r > \gamma$.

Proof of Theorem 2.1. Fix $r > \gamma$.

$$(1-A)^{-r}x = \frac{1}{\Gamma(r)}\int_0^\infty e^{-u}u^{r-1}e^{uA}x\,du$$

[10, Proposition 11.1], thus for $z = t+is$,

$$(1-A)^{-r}e^{zA}x = \frac{1}{\Gamma(r)}\int_0^\infty e^{-u}u^{r-1}e^{((t+u)+is)A}x\,du,$$

so that we may estimate as follows.

$$(*)\qquad \begin{aligned}\|(1-A)^{-r}e^{zA}\| &\le \frac{M}{\Gamma(r)}\int_0^\infty e^{-u}u^{r-1}\left(\frac{1}{u+t}\right)^\gamma\left(\sqrt{(u+t)^2+s^2}\right)^\gamma du\\ &\le \frac{M}{\Gamma(r)}\int_0^\infty e^{-u}u^{r-1-\gamma}\left(\sqrt{(u+t)^2+s^2}\right)^\gamma du,\end{aligned}$$

which is convergent since $r-1-\gamma > -1$.

As with strongly continuous holomorphic semigroups, since $(1-A)^{-r}e^{zA}$ is holomorphic and bounded in every rectangle $\{t+is \mid 0 < t < 1,\ |s| < a\}$, $a > 0$, its boundary values exist when $t \to 0$ and define a $(1-A)^{-r}$-regularized group $\{W_r(s)\}_{s\in\mathbf{R}}$. It is straightforward to verify that the generator of the $(1-A)^{-r}$-regularized semigroup $\{(1-A)^{-r}e^{zA}\}_{z\in\mathrm{RHP}}$ is A, thus iA generates $\{W_r(s)\}_{s\in\mathbf{R}}$.

All that remains is the growth estimate on $\|W_r(s)\| = \lim_{t\to 0}\|(1-A)^{-r}e^{(t+is)A}\|$, which by $(*)$ is less than or equal to

$$\frac{M}{\Gamma(r)}\int_0^\infty e^{-u}u^{r-1-\gamma}\left(\sqrt{u^2+s^2}\right)^\gamma du.$$

The integral may be shown to be less than or equal to

$$(1+|s|^\gamma)\int_0^\infty e^{-u}u^{r-1-\gamma}\left(\sqrt{1+u^2}\right)^\gamma du,$$

by considering separately $|s|\le 1$ and $|s|\ge 1$, for if $|s|\le 1$ then $u^2+s^2\le u^2+1$, while if $|s|\ge 1$ then

$$u^2+s^2=s^2((u/s)^2+1)\le s^2(u^2+1). \quad \square$$

Proof of Theorem 2.2. (a) $\rightarrow$ (b) and (c) $\rightarrow$ (b) are essentially the same as the proof of Theorem 2.1.

(b) $\rightarrow$ (c) is a consequence of the maximum principle for analytic functions.

(b) $\rightarrow$ (a). Fix $r>\gamma$. Since $\{e^{zA}\}$ is a bounded strongly continuous holomorphic semigroup, $\exists K_r<\infty$ such that

$$\|(1-A)^r e^{tA}\|\le K_r t^{-r} \quad \forall t>0$$

(see [12, last section of Chapter 2]).

For $z=x+iy$, with $x>0$,

$$\begin{aligned}\|e^{zA}\| &= \|(1-A)^r e^{xA}W_r(y)\|\le \|(1-A)^r e^{xA}\|\,\|W_r(y)\| \\ &\le K_r x^{-r}C_r(1+|y|^r)\le (K_r)(C_r)\left(\frac{1+|z|}{\mathrm{Re}(z)}\right)^r. \quad \square\end{aligned}$$

III. Application to the Schrödinger operator

We show how Theorem 2.1 may be applied to the Schrödinger operator $i(\Delta-V)$ for appropriate potentials V.

Theorem 3.1. *Suppose* $1\le p<\infty$. *Then* $\forall r>n|\frac{1}{p}-\frac{1}{2}|$, $i\Delta$ *on* $L^p(\mathbf{R}^n)$ *generates a* $(1-\Delta)^{-r}$*-regularized group* $\{W_r(s)\}_{s\in\mathbf{R}}$ *that is* $O((1+|s|^{n|1/p-1/2|}))$.

On $C_0(\mathbf{R}^n)$ *or* $BUC(\mathbf{R}^n)$, $i\Delta$ *generates a* $(1-\Delta)^{-r}$*-regularized group* $\{W_r(s)\}_{s\in\mathbf{R}}$ *that is* $O(1+|s|^{n/2})$ $\forall r>n/2$.

Definition 3.2. We will denote by K^n the *Kato class* of measurable functions on $\mathbf{R}^n$ as defined in [14, p. 453]. This includes, but is not limited to $L^\infty(\mathbf{R}^n)$.

Definition 3.3. For $V\in K^n$, it is shown in [14, Theorem A.2.7] that $H\equiv\Delta-V$, defined as a quadratic form, is a selfadjoint operator on $L^2(\mathbf{R}^n)$.

The following theorem states that H, with appropriate domain, generates an exponentially bounded $(\omega-H)^{-r}$-regularized group for r twice as big as in Theorem 3.1.

Theorem 3.4. *Suppose* $1\le p<\infty$, $V_+\in K^n$, *and* $V_-\in L^\infty(\mathbf{R}^n)$. *Let* H *be as in Definition* 3.3. *Then* $\exists\omega\in\mathbf{R}$ *such that* $\forall r>2n|1/p-1/2|$, $\{e^{isH}(\omega-H)^{-r}\}_{s\in\mathbf{R}}$ *is an* $(\omega-H)^{-r}$*-regularized group on* $L^p(\mathbf{R}^n)$ *that is* $O((1+|s|^{2n|1/p-1/2|}))$.

On $C_0(\mathbf{R}^n)$ *or* $BUC(\mathbf{R}^n)$, $\{e^{isH}(\omega-H)^{-r}\}_{s\in\mathbf{R}}$ *is an* $(\omega-H)^{-r}$*-regularized group that is* $O((1+|s|^{n/2}))$ $\forall r>n$.

Theorem 3.1 follows immediately from Theorem 2.1 and the following lemma.

Lemma 3.5. *Suppose* $1 \leq p \leq \infty$ *and* $n \in \mathbf{N}$. *Then*

$$\|e^{z\Delta}\|_p \leq \left(\frac{|z|}{\mathrm{Re}(z)}\right)^{n|1/p-1/2|},$$

whenever $\mathrm{Re}(z) > 0$.

Proof. The heat semigroup is a convolution operator with kernel $K_z(x) \equiv (4\pi z)^{-n/2}e^{-x^2/4z}$, that is,

$$e^{z\Delta}f = K_z * f,$$

$\forall f \in L^p(\mathbf{R}^n)$, $\mathrm{Re}(z) > 0$. Thus a direct computation shows that

$$\|e^{z\Delta}\|_1 = \|K_z\|_1 = \left(\frac{|z|}{\mathrm{Re}(z)}\right)^{n/2}.$$

It is also well known that $\|e^{z\Delta}\|_2 = 1$ whenever $\mathrm{Re}(z) > 0$.

Fix z in the open right half-plane. We now use the Riesz Convexity Theorem [7, p. 523], which asserts that $h(a) \equiv \log\|e^{z\Delta}\|_{1/a}$ is a convex function of a in $[0, 1]$. First consider $1 \leq p \leq 2$, and set $p = \frac{1}{a}$, so that a is in the interval $[\frac{1}{2}, 1]$. For such p, the convexity of h implies that

$$\log\|e^{z\Delta}\|_p \leq \alpha \log\|e^{z\Delta}\|_2 + \beta \log\|e^{z\Delta}\|_1 = \beta \log\|e^{z\Delta}\|_1,$$

where $\alpha+\beta = 1$, $\frac{1}{2}\alpha+\beta = a = \frac{1}{p}$. The last two equations imply that $\beta = \frac{2}{p} - 1$, thus

$$\|e^{z\Delta}\|_p \leq \|e^{z\Delta}\|_1^{2/p-1} = \left(\frac{|z|}{\mathrm{Re}(z)}\right)^{n(1/p-1/2)},$$

proving the lemma when $1 \leq p \leq 2$.

For $2 \leq p \leq \infty$, duality implies that, if $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\|e^{z\Delta}\|_p = \|e^{z\Delta}\|_q \leq \left(\frac{|z|}{\mathrm{Re}(z)}\right)^{n(1/q-1/2)} = \left(\frac{|z|}{\mathrm{Re}(z)}\right)^{n(1/2-1/p)},$$

concluding the proof. □

For the proof of Theorem 3.4, we will need the following from [14, Theorem B.7.1; 4, Theorem 9]; see also [11, Propositions 2.1 and 2.4].

Lemma 3.6. *Let* H *be as in Definition* 3.3. *Then* $\exists \mu, c, a \in \mathbf{R}^+$ *and a kernel* $\widetilde{K}(z, x, y)$ *such that*

$$e^{z(H-\mu)}f(x) = \int_{\mathbf{R}^n} \widetilde{K}(z, x, y)f(y)\,dy$$

for $\mathrm{Re}(z) > 0$, $f \in L^p(\mathbf{R}^n)$ $(1 \leq p \leq \infty)$ *and*

$$|\widetilde{K}(z, x, y)| \leq c(\mathrm{Re}(z))^{-n/2} \exp\left(-\mathrm{Re}\left(\frac{|x-y|^2}{az}\right)\right)$$

for $\mathrm{Re}(z) > 0$, $x, y \in \mathbf{R}^n$.

Proof of Theorem 3.4. Let $\omega \equiv \max\{\|V_-\|_\infty, \mu\} + 1$. We argue as in Theorem 3.1 with Δ replaced by $H + 1 - \omega$.

Since $V+\omega-1$ is a real-valued nonnegative Kato perturbation so that $-z(V+\omega-1)$ is a dissipative Kato perturbation, $\|e^{z(H+1-\omega)}\|_2 \leq 1 \ \ \forall \operatorname{Re}(z) > 0$ (see [8, Corollary 6.8]).

By Lemma 3.6,

$$\|e^{z(H+1-\omega)}\|_1 \leq \left\|(\operatorname{Re}(z))^{-n/2} \exp\left(-\operatorname{Re}\left(\frac{|x|^2}{az}\right)\right)\right\|_1 \leq c'\left(\frac{|z|}{\operatorname{Re}(z)}\right)^n,$$

where c' is independent of z, so that we may argue exactly as in the proof of Theorem 3.1 to conclude that $\forall r > 2n|\frac{1}{p}-\frac{1}{2}|$,

$$\{e^{isH}(\omega-H)^{-r}\} = \{e^{is(\omega-1)}e^{is(H+1-\omega)}(1-(H+1-\omega))^{-r}\}$$

is a $O((1+|s|^{2n|1/p-1/2|}))(\omega-H)^{-r}$-regularized semigroup. □

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Department of Mathematics and Computer Science, Ohio Northern University, Meyer Hall, Ada, Ohio 45810–1599

Department of Mathematics, Ohio University, Athens, Ohio 45701