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HOROCYCLES ON RIEMANN SURFACES

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ABSTRACT. By the Collar Theorem, every puncture on a hyperbolic Riemann surface with punctures has a horocyclic neighborhood of area 2. Furthermore two such neighborhoods associated to different punctures are disjoint.

This result can be improved if we omit the condition that horocyclic neighborhoods of different punctures must be disjoint. Using arguments of the second author we show, in this paper, that each puncture of a hyperbolic Riemann surface has a horocyclic neighborhood of area 4.

1. PRELIMINARIES

We consider hyperbolic Riemann surfaces X of finite type. By the Uniformization we may express X as U/G, where U is the upper half-plane and G is a Fuchsian group. The hyperbolic metric, given by the line element $|dz|/\mathrm{Im}\, z$ induces, on the Riemann surface X, a metric of constant curvature -1.

If X is not compact, then it has either ideal boundary components or punctures (or both). We focus our attention on punctures of X here. They correspond to the fixed-points of parabolic elements of the group G.

Let p be a puncture of X. By conjugation we may assume that the parabolic element corresponding to p is $g_\omega(z) = z + \omega$. (By an additional conjugation we could even assume that $\omega = 1$ here.)

By saying that g_ω *corresponds* to p we implicitly assume also that g_ω is *primitive*, i.e., that it satisfies the following condition:

$$\exists h \in G \wedge \exists n, m \in \mathbf{Z}: \; g_\omega^n = h^m \Rightarrow n/m \in \mathbf{Z}.$$

For $\lambda > 0$ let $H_\lambda = \{z \mid \mathrm{Im}\, z > \lambda\}$. Assume that λ is such that the following holds:

$$g(H_\lambda) \cap H_\lambda \neq \varnothing \Rightarrow g \in \langle g_\omega \rangle, \tag{1}$$

where $\langle g_\omega \rangle$ is the group generated by the transformation g_ω.

Condition (1) implies that H_λ projects onto a punctured disk on X that is a neighborhood of the puncture p. Such a set is called a *horocyclic neighborhood* of the puncture p.

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The hyperbolic area of this horocyclic neighborhood of p equals the hyperbolic area of the infinite half-strip

$$\{z \mid \operatorname{Im} z > \lambda, \ 0 < \operatorname{Re} z < \omega\},$$

which is ω/λ.

2. Large horocyclic neighborhoods of punctures

In the case we are now considering, G is a Fuchsian group that acts freely in the upper half-plane U. This means that the group G does not contain elliptic elements. Let $g \in G$, $g(\infty) \neq \infty$, and let $I(g)$ denote the isometric circle of g. The center of $I(g)$ lies on the real axis, $g(I(g)) = I(g^{-1})$, and $I(g)$ and $I(g^{-1})$ have the same radius. If g is parabolic, then $I(g)$ and $I(g^{-1})$ are tangent to each other at the fixed point of g, otherwise $I(g) \cap I(g^{-1}) = \varnothing$.

Lemma 2.1. *The difference $g(z) - z$ is real for a point z in the upper half-plane if and only if $z \in I(g)$.*

This lemma follows directly from the geometry of the action of the Möbius transformation g and the definition of the isometric circle. The proof is left to the reader.

Lemma 2.2. *Let $g \in G$ be such that $g(\infty) \neq \infty$. If G contains the translation $g_\omega : z \mapsto z + \omega$, then $(g(z) - z)/\omega$ is not an integer for any $z \in U$.*

Proof. Suppose that $g(z) - z = n\omega$ for some integer n and $z \in U$. Then $g^{-1} \circ g_\omega^n$ fixes z. It follows that $g^{-1} \circ g_\omega^n$ is elliptic, which is not possible by our assumptions. □

Lemma 2.3. *Suppose that $g_\omega : z \mapsto z + \omega$ is in G. If $g \in G$ does not fix ∞, then the radius r_g of $I(g)$ satisfies $r_g \leq \omega/4$.*

Proof. By geometry

$$\max_{z \in I(g)} |g(z) - z| - \min_{z \in I(g)} |g(z) - z| = 4r_g. \tag{2}$$

Now $(g(z) - z)/\omega$ is real on $I(g)$ by Lemma 2.1. If the total variation of $(g(z) - z)/\omega$ along $I(g)$ were more than 1, then $(g(z) - z)/\omega$ would necessarily take an integer value at some point in $I(g)$. By Lemma 2.2 this is not possible. We conclude, therefore, that

$$\max_{z \in I(g)} \frac{|g(z) - z|}{\omega} - \min_{z \in I(g)} \frac{|g(z) - z|}{\omega} \leq 1. \tag{3}$$

Inequality (3) together with equation (2) now implies the lemma. □

Theorem 2.4. *Assume that the group G contains, besides the identity, only hyperbolic and parabolic Möbius transformations mapping the upper half-plane onto itself. Assume further that $g_1(z) = z + \omega$ is a primitive element of the group G. Let $g \in G$ be such that $g(\infty) \neq \infty$. If $\operatorname{Im} z > \omega/4$, then $\operatorname{Im} g(z) < \omega/4$. The number $\omega/4$ is the smallest possible.*

Proof. The first part of the statement follows directly from Lemma 2.3. To prove that $\omega/4$ is the smallest number with this property, assume that $\omega = 1$ and let g_0 be the transformation $z \mapsto z/(4z + 1)$ and G_1 be the group generated by g_0 and g_1. Then G_1 is a Fuchsian group without elliptic elements and $g_0((-1+i)/4) = (1+i)/4$. $\square$

Theorem 2.4 now immediately implies the following result:

Theorem 2.5. *Let X be a hyperbolic Riemann surface with punctures. Each puncture of X has a horocyclic neighborhood of area* 4. *The inner boundary curve of this horocycle has length* 4.

Observe that area-4 horocyclic neighborhoods of punctures need not be disjoint. The group G_1 provides an example of this: the area-4 horocycle of g_1 is the half-plane $\operatorname{Im} z > 1/4$ while the area-4 horocycle for g_0 is the open Euclidean disk of radius $1/2$ and center $i/2$. They overlap. Observe also that simple closed geodesics can enter area-4 horocyclic neighborhoods of punctures. We thank the referee for this observation.

By the above considerations, it is obvious that we can always find *disjoint* area-2 horocycles at punctures. Simple closed curves do not enter these area-2 horocyclic neighborhoods. But this, on the other hand, is already well known (see, e.g., [4, 1.2, pp. 507–508]).

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