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A NOTE ON THE MACKEY DUAL OF $C(K)$

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ABSTRACT. Let K be a compact metric space, and let τ denote the Mackey topology on $M(K)$ with respect to the $\langle C(K), M(K)\rangle$ duality. That is, τ is the topology of uniform convergence on the weakly compact subsets of $C(K)$. Just as for the weak* topology, the dual space of $(M(K), \tau)$ is $C(K)$. However, τ is very different from weak*. Indeed, it is obvious that if $\{x_n\}$ is a sequence converging to x in K, then $\delta(x_n)$ converges to $\delta(x)$ in the weak* topology, yet Kirk has shown (Pacific J. Math. **45** (1973), 543–554) that $\{\delta(x) \mid x \in K\}$ is closed and discrete in the Mackey topology. We obtain a further result along these lines: For each $A \subset K$ set $\Delta A = \{\delta(x) - \delta(y) \mid x \neq y, x, y \in A\}$. Let $\mathscr{D}$ denote the totality of all subsets A of K with the property that $0 \in \overline{\Delta A}^{\tau}$. Then a closed set is in $\mathscr{D}$ iff it is uncountable. Alternatively stated, a closed subset A of K is countable if and only if there is a weakly compact subset L of $C(K)$ such that for every pair $x, y \in A$, $x \neq y$, there is an $h \in L$ with $|h(x) - h(y)| \geq 1$.

Throughout, K is a compact metric space.

Our interest here is in the duality between the space of continuous, real-valued functions on K, $C(K)$, and its Banach space dual, the regular Borel measures on K, which we denote by $M(K)$. Much, of course, is known about this duality, however, the bulk of this knowledge is with respect to the weak and weak* topologies. We consider the Mackey topology on $M(K)$, $\tau := \tau(M(K), C(K))$, which is the topology of uniform convergence on the weakly compact subsets of $C(K)$. The first notable application of the Mackey topology to the study of Banach spaces was made by Grothendieck [1], who showed the connection between τ-compacta and the Dunford-Pettis (DP) and reciprocal Dunford-Pettis (RDP) properties. For the case of $C(K)$, which has both DP and RDP, Grothendieck's results tell us that the τ-compact and $\sigma(M(K), M(K)^*)$-compact sets are identical. Other studies involving the Mackey dual of a Banach space have been published more recently [3, 5, 6].

We denote the closed unit ball of $C(K)$ by $B_{C(K)}$, and for an element $f \in C(K)$, the support of f is defined to be $\operatorname{supp}(f) := \{x \in K \mid f(x) \neq 0\}$. Given a subset A of K and an ordinal γ, we denote the γth derived set of A by

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$A^{(\gamma)}$. For each $A \subset K$, let

$$\Delta A = \{\delta(x) - \delta(y) \in M(K) \mid x \neq y \text{ and } x, y \in A\},$$

and let $\mathscr{D}$ denote the collection of all subsets, A, of K with the property that 0 is in the τ-closure of ΔA. It is equivalent to require that for each weakly compact subset L of $C(K)$, there exist $x, y \in A$, $x \neq y$, with $|h(x) - h(y)| < 1\ \forall h \in L$.

In the proof of the following lemma, only properties (1) and (4) are used. However, properties (2), (3), and (5) are important in the proof of Theorem 2 and are stated in the lemma solely to make the proof of the theorem more coherent.

Lemma 1. *Let A be a closed subset of K and let γ be an ordinal number. Suppose that for each $\varepsilon > 0$, there is an $H \subset B_{C(K)}$ with the following properties:*

(1) *H is relatively weakly compact.*
(2) *Each element of H is supported on an open ball of diameter less than ε.*
(3) *Each $h \in H$ is 0 on the γth derived set of A, $A^{(\gamma)}$.*
(4) *For every $\beta < \gamma$ and $x \in A^{(\beta)} \backslash A^{(\beta+1)}$, there is an $h \in H$ such that $h(x) = 1$ and $\operatorname{supp}(h) \cap A^{(\beta)} = \{x\}$.*
(5) *For each $h \in H$, there is an $x \in A \backslash A^{(\gamma)}$ such that $h(x) = 1$.*

Then $A \in \mathscr{D} \Rightarrow A^{(\gamma)} \in \mathscr{D}$.

Proof. Suppose $A \in \mathscr{D}$ and $H_1 \subset C(K)$ is weakly compact. Fix $\varepsilon > 0$ and choose H as above. By assumption, there exists $x, y \in A$ with $x \neq y$ such that $|h(x) - h(y)| < 1$ for all $h \in H_1 \cup H$. If $\{x, y\} \subset A^{(\gamma)}$, we are done, so suppose that this is not the case. Then, without loss of generality, there is a $\beta < \gamma$ such that $x \in A^{(\beta)} \backslash A^{(\beta+1)}$ and $y \in A^{(\beta)}$. By condition (4), there is an $h \in H$ such that $h(x) = 1$ and $h(y) = 0$. This is a contradiction, so we must have $\{x, y\} \subset A^{(\gamma)}$, hence $A^{(\gamma)} \in \mathscr{D}$. □

Theorem 2. *If A is a closed subset of K and $A \in \mathscr{D}$, then $A^{(\alpha)} \in \mathscr{D}$ for every countable ordinal α.*

Proof. We show by transfinite induction that for any countable ordinal α and any $\varepsilon > 0$, there is an $H \subset B_{C(K)}$ that satisfies conditions (1)–(5) of Lemma 1 for $\gamma = \alpha$.

Suppose $\alpha = 1$. For each $x \in A \backslash A^{(1)}$, find an open ball B_x of diameter less than ε, containing x, such that $B_x \cap A^{(1)} = \varnothing$, and so that $B_x \cap B_y = \varnothing$ for all $x \neq y \in A \backslash A^{(1)}$. Now find, for each $x \in A \backslash A^{(1)}$, an $h_x \in C(K)$ supported in B_x with $h_x(x) = 1$ and $0 \leq h_x \leq 1$. Then $\{h_x \mid x \in A \backslash A^{(1)}\}$ clearly satisfies conditions (2)–(5) of Lemma 1 for $\gamma = \alpha = 1$, and (1) holds since any uniformly bounded collection of continuous functions with pairwise disjoint supports is relatively weakly compact.

Now let α be a countable ordinal with the property that for every $\beta < \alpha$ and every $\varepsilon > 0$, there is an $H_{\beta, \varepsilon} \subset B_{C(K)}$ satisfying conditions (1)–(5) for $\gamma = \beta$. We will show that these conditions hold for $\gamma = \alpha$ as well.

First, suppose that α is not a limit ordinal, say $\alpha = \beta + 1$, for some ordinal β. By the inductive hypotheses, find $H_1 \subset B_{C(K)}$ satisfying conditions (1)–(5) for $\gamma = \beta$. For each $x \in A^{(\beta)} \backslash A^{(\alpha)}$, find an open ball B_x of diameter less than

ε, containing x, such that $B_x \cap A^{(\alpha)} = \varnothing$, and so that $B_x \cap B_y = \varnothing$ for all $x \neq y \in A^{(\beta)} \backslash A^{(\alpha)}$. Now find, for each $x \in A^{(\beta)} \backslash A^{(\alpha)}$, an $h_x \in C(K)$ supported in B_x with $h(x) = 1$ and $0 \le h_x \le 1$. Then, letting $H_2 = \{h_x \mid x \in A^{(\beta)} \backslash A^{(\alpha)}\}$ and setting $H_{\alpha,\varepsilon} = H_1 \cup H_2$, we immediately have that $H_{\alpha,\varepsilon}$ satisfies conditions (2)–(5) for $\gamma = \alpha$. To see that $H_{\alpha,\varepsilon}$ also obeys (1) for $\gamma = \alpha$, consider that this is true separately for H_1 and H_2; H_1 by inductive hypothesis and H_2 by the pairwise disjointness of the supports of its elements. Thus, the induction is established for nonlimit ordinals.

Now let us assume that α is a limit ordinal. Let $\{\beta_i\}_{i \ge 1}$ be such that $0 = \beta_0 < \beta_1 < \cdots < \alpha$ and $\lim_{i\to\infty} \beta_i = \alpha$. For each $i \ge 1$, let $H_{\beta_i, \delta_i} \subset B_{C(K)}$ satisfy conditions (1)–(5) of the lemma with ε replaced by $\delta_i = \min\{\varepsilon, 1/i\}$ and with $\gamma = \beta_i$. Also for each $i \ge 1$, set

$$F_i = \{f \in H_{\beta_i, \delta_i} \mid f(x) = 1, \text{ for some } x \in A^{(\beta_{i-1})}\},$$

and let $H_{\alpha,\varepsilon} = \bigcup_{i \ge 1} F_i$.

We claim that $H_{\alpha,\varepsilon}$ satisfies (1)–(5) for ε and $\gamma = \alpha$. Let us first show that (1) is satisfied. Suppose $\{g_n\}_{n \ge 1}$ is a sequence in $H_{\alpha,\varepsilon}$ and does not have a pointwise convergent subsequence. Then, since the F_i are relatively weakly compact, we may assume, without loss of generality, that there is an increasing sequence of positive integers, $\{m_n\}_{n \ge 1}$, such that $g_n \in F_{m_n}$ for all $n \ge 1$. Suppose that there is an $x \in K$ such that $g_n(x)$ is not eventually 0. Then, without loss of generality, we assume that $g_n(x)$ is never 0. Now for each n there is an $x_n \in A^{(\beta_{m_n} - 1)} \backslash A^{(\alpha)}$ such that $g_n(x_n) = 1$ since $g_n \in F_{m_n}$. But then $|x - x_n| < 1/n$ for each n, so $x_n \to x$, and hence $x \in A^{(\alpha)}$. This is a contradiction, since all g_n are 0 on $A^{(\alpha)}$. Therefore, $H_{\alpha,\varepsilon}$ satisfies (1).

Condition (2) clearly holds, and so does (3), since $h \in H_{\alpha,\varepsilon}$ implies $h \in F_i$ for some i, so $h = 0$ on $A^{(\beta_i)} \supset A^{(\alpha)}$. Condition (5) holds similarly, since $h \in F_i$ implies that there is an $x \in A^{(\beta_{i-1})}$ such that $h(x) = 1$, and by (3), $x \in A \backslash A^{(\alpha)}$. To see that condition (4) is satisfied, let $\beta < \alpha$ and $x \in A^{(\beta)} \backslash A^{(\beta+1)}$. Then there is an i such that $\beta_i < \beta \le \beta_{i+1}$, and hence, by the choice of F_{i+1} (namely, that condition (4) is satisfied with $\gamma = \beta_{i+1}$) there is an $h \in F_{i+1} \subset H_{\alpha,\varepsilon}$ such that $h(x) = 1$ and the support of x meets $A^{(\beta)}$ only at x. □

Corollary 3. *A closed subset A of a compact metric space K is a member of $\mathscr{D}$ iff A is uncountable.*

Proof. ($\Rightarrow$) If A does not contain a perfect subset then A is countable by the Cantor-Bendixson Theorem [2, p. 72, 6.66], hence $A^{(\alpha)} = \varnothing$ for some countable ordinal α. Thus, by Theorem 2, $A \notin \mathscr{D}$.

($\Leftarrow$) Without loss of generality, assume that A is perfect. Let H be a weakly compact subset of $C(K)$. Then $\{h|_A \mid h \in H\}$ is a weakly compact subset of $C(A)$, and hence $\{h|_A \mid h \in H\}$ is equicontinuous at each point of a dense subset D of A [4, p. 522, 2.4]. Let $x \in D$, and find $\delta > 0$ such that if $y \in A \cap B_\delta$, where B_δ is the open ball of radius δ centered at x, then $|h(x) - h(y)| < \frac{1}{2}$ for every $h \in H$. Since A is perfect, we can now find a $y, z \in A \cap B_\delta$, with $y \neq z$, hence $|h(y) - h(z)| < 1$ for all $h \in H$. Therefore, $A \in \mathscr{D}$. □

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