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ON CLOSED SUBSPACES OF OPERATOR RANGES

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ABSTRACT. Necessary and sufficient for the closure of a linear subspace to lie in the range of a bounded linear operator is a certain "bounded preimage property" for the operator.

If $T: X \to Y$ is a bounded linear operator between normed spaces then we shall, *par abus de notation*, also write [3]

$$T: l_\infty(X) \to l_\infty(Y) \tag{0.1}$$

for the operator induced between the corresponding spaces of bounded vector-valued sequences

$$l_\infty(X) = \left\{x \in X^{\mathbb{N}}: \sup_n \|x_n\| < \infty\right\}. \tag{0.2}$$

1. **Theorem.** *If $T \in \mathrm{BL}(X, Y)$ is a bounded linear operator between Banach spaces and if $M \subseteq Y$ is a linear subspace, then there is equivalence*

$$\mathrm{cl}\, M \subseteq T(X) \Leftrightarrow l_\infty(M) \subseteq T l_\infty(X). \tag{1.1}$$

Proof. We shall show forward implication for complete X and backward implication for complete Y. Whether or not either space is complete, the right-hand side of (1.1) is equivalent to

$$T_M^\wedge: T^{-1}(M)/T^{-1}(0) \to Y \text{ bounded below.} \tag{1.2}$$

Indeed if (1.2) holds then there is $k > 0$ for which

$$\mathrm{dist}(x, T^{-1}(0)) \le k\|Tx\| \quad \text{for each } x \in T^{-1}(M),$$

so that if $y \in l_\infty(M)$ is arbitrary then there is $x \in X^{\mathbb{N}}$ for which

$$y = Tx \quad \text{with } \mathrm{dist}(x_n, T^{-1}(0)) \le k\|y_n\|,$$

and then $z \in T^{-1}(0)^{\mathbb{N}}$ for which

$$\|x - z\| \le 2\, \mathrm{dist}(x, T^{-1}(0)),$$

giving

$$y = T(x - z) \quad \text{with } x - z \in l_\infty(X). \tag{1.3}$$

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Conversely if (1.2) fails then there is $x \in X^{\mathbb{N}}$ for which

$$Tx_n \in M, \quad \|Tx_n\| \to 0, \quad \operatorname{dist}(x_n, T^{-1}(0)) \geq 1.$$

Now with

$$x'_n = \begin{cases} \|Tx_n\|^{-1/2}x_n & \text{if } Tx_n \neq 0, \\ nx_n & \text{if } Tx_n = 0, \end{cases}$$

we have $\|Tx'_n\| \to 0$ and $\operatorname{dist}(x'_n, T^{-1}(0)) \to \infty$ so that

(1.4) $$Tx' \in c_0(M) \subseteq l_\infty(M) \quad \text{and} \quad Tx' \notin Tl_\infty(X).$$

If, in particular, the spaces X and Y are complete then condition (1.2) is also equivalent to the left-hand side of (1.1). To see this we need an auxiliary subspace

(1.5) $$M^\sim = T\operatorname{cl} T^{-1}(M).$$

Evidently

(1.6) $$M \subseteq M^\sim \subseteq T(X) \cap \operatorname{cl} M,$$

and hence, in particular,

(1.7) $$T_M^\wedge \text{ bounded below} \Leftrightarrow T_{M^\sim}^\wedge \text{ bounded below.}$$

The operator $T_{M^\sim}^\wedge$ is one-to-one, with range $M^\sim$, and if X is complete defined on the complete space

(1.8) $$T^{-1}(M^\sim)/T^{-1}(0) = \operatorname{cl} T^{-1}(M)/T^{-1}(0),$$

so that

(1.9) $$T_{M^\sim}^\wedge \text{ bounded below} \Rightarrow M^\sim = \operatorname{cl} M^\sim$$

since $M^\sim$ is complete. By (1.6) this gives

(1.10) $$M^\sim = \operatorname{cl} M,$$

and hence also the left-hand side of (1.1) holds. Conversely if this happens then $\operatorname{cl} M$ is complete (if Y is) and the open mapping theorem gives

(1.11) $$T_{\operatorname{cl} M}^\wedge : T^{-1}(\operatorname{cl} M)/T^{-1}(0) \to Y \text{ bounded below,}$$

and hence also (1.2). □

The same argument gives the analogue of Theorem 1 in which the right-hand side of (1.1) is replaced by the corresponding property for subsets

(1.12) $$\beta(M) \subseteq T\beta(X),$$

where $\beta(X)$ denotes the bounded subsets of X; an easy consequence is that compact operators on complete spaces have the "Calkin property" [4; 2, Theorem III.1.12]

(1.13) $$\operatorname{cl} M \subseteq T(X) \Rightarrow M \text{ finite dimensional.}$$

Notice that we have proved two versions of Theorem 1: we also have

(1.14) $$\operatorname{cl} M \subseteq T(X) \Leftrightarrow c_0(M) \subseteq Tl_\infty(X).$$

In the particular case $M = T(X)$ Albrecht and Mehta [1, Lemma 2.1] have shown that also

(1.15) $$\operatorname{cl} M \subseteq T(X) \Leftrightarrow l_\infty(M) \subseteq T(X) + c_0(Y),$$

which says that the image of M in the "enlargement" of Y [3, Definition 1.9.2] is included in the range of the enlargement of T.

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