ON CLOSED SUBSPACES OF OPERATOR RANGES

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Abstract. Necessary and sufficient for the closure of a linear subspace to lie in the range of a bounded linear operator is a certain "bounded preimage property" for the operator.

If $T: X \to Y$ is a bounded linear operator between normed spaces then we shall, par abus de notation, also write [3]

(0.1) $T: l_\infty(X) \to l_\infty(Y)$

for the operator induced between the corresponding spaces of bounded vector-valued sequences

(0.2) $l_\infty(X) = \left\{ x \in X^\mathbb{N}: \sup_n \|x_n\| < \infty \right\}.$

1. Theorem. If $T \in \operatorname{BL}(X, Y)$ is a bounded linear operator between Banach spaces and if $M \subseteq Y$ is a linear subspace, then there is equivalence

(1.1) $\operatorname{cl} M \subseteq T(X) \iff l_\infty(M) \subseteq Tl_\infty(X).$

Proof. We shall show forward implication for complete $X$ and backward implication for complete $Y.$ Whether or not either space is complete, the right-hand side of (1.1) is equivalent to

(1.2) $T(M)_T: T^{-1}(M)/T^{-1}(0) \to Y$ bounded below.

Indeed if (1.2) holds then there is $k > 0$ for which

$$\operatorname{dist}(x, T^{-1}(0)) \leq k\|Tx\| \quad \text{for each } x \in T^{-1}(M),$$

so that if $y \in l_\infty(M)$ is arbitrary then there is $x \in X^\mathbb{N}$ for which

$$y = Tx \quad \text{with } \operatorname{dist}(x_n, T^{-1}(0)) \leq k\|y_n\|,$$

and then $z \in T^{-1}(0)^\mathbb{N}$ for which

$$\|x - z\| \leq 2 \operatorname{dist}(x, T^{-1}(0)),$$

giving

(1.3) $y = T(x - z) \quad \text{with } x - z \in l_\infty(X).$

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Conversely if (1.2) fails then there is $x \in X^N$ for which

$$Tx_n \in M, \quad \|Tx_n\| \to 0, \quad \text{dist}(x_n, T^{-1}(0)) \geq 1.$$  

Now with

$$x'_n = \begin{cases} \|Tx_n\|^{-1/2}x_n & \text{if } Tx_n \neq 0, \\ nx_n & \text{if } Tx_n = 0, \end{cases}$$

we have $\|Tx'_n\| \to 0$ and $\text{dist}(x'_n, T^{-1}(0)) \to \infty$ so that

$$(1.4) \quad Tx' \in c_0(M) \subseteq l_\infty(M) \quad \text{and} \quad Tx' \not\subseteq Tl_\infty(X).$$

If, in particular, the spaces $X$ and $Y$ are complete then condition (1.2) is also equivalent to the left-hand side of (1.1). To see this we need an auxiliary subspace

$$(1.5) \quad M^\sim = T\text{cl } T^{-1}(M).$$

Evidently

$$(1.6) \quad M \subseteq M^\sim \subseteq T(X) \cap \text{cl } M,$$

and hence, in particular,

$$(1.7) \quad T_{M^\sim}^\wedge \text{bounded below } \Leftrightarrow T_{M^\sim}^\wedge \text{bounded below}.$$  

The operator $T_{M^\sim}^\wedge$ is one-to-one, with range $M^\sim$, and if $X$ is complete defined on the complete space

$$T^{-1}(M^\sim)/T^{-1}(0) = \text{cl } T^{-1}(M)/T^{-1}(0),$$

so that

$$T_{M^\sim}^\wedge \text{bounded below } \Rightarrow M^\sim = \text{cl } M^\sim$$

since $M^\sim$ is complete. By (1.6) this gives

$$M^\sim = \text{cl } M,$$

and hence also the left-hand side of (1.1) holds. Conversely if this happens then $\text{cl } M$ is complete (if $Y$ is) and the open mapping theorem gives

$$(1.11) \quad T_{\text{cl } M}^\wedge: T^{-1}(\text{cl } M)/T^{-1}(0) \to Y \text{ bounded below},$$

and hence also (1.2).  \qed

The same argument gives the analogue of Theorem 1 in which the right-hand side of (1.1) is replaced by the corresponding property for subsets

$$(1.12) \quad \beta(M) \subseteq T\beta(X),$$

where $\beta(X)$ denotes the bounded subsets of $X$; an easy consequence is that compact operators on complete spaces have the “Calkin property” [4; 2, Theorem III.1.12]

$$(1.13) \quad \text{cl } M \subseteq T(X) \Rightarrow M \text{ finite dimensional}.$$  

Notice that we have proved two versions of Theorem 1: we also have

$$(1.14) \quad \text{cl } M \subseteq T(X) \Leftrightarrow c_0(M) \subseteq Tl_\infty(X).$$

In the particular case $M = T(X)$ Albrecht and Mehta [1, Lemma 2.1] have shown that also

$$(1.15) \quad \text{cl } M \subseteq T(X) \Leftrightarrow l_\infty(M) \subseteq T(X) + c_0(Y),$$

which says that the image of $M$ in the “enlargement” of $Y$ [3, Definition 1.9.2] is included in the range of the enlargement of $T$.  \vspace{0.5cm}
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