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A REMARK ON O'HARA'S ENERGY OF KNOTS

NOBUMITSU NAKAUCHI

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ABSTRACT. We show a relation between O'Hara's energy of knots and the Douglas functional.

In his paper [4], O'Hara introduced the following energy of knots

$$E_{\text{knot}}(\varphi) := \frac{1}{2}\int\int_{S^1\times S^1}\left\{\frac{1}{|\varphi(x)-\varphi(y)|^2} - \frac{\pi^2}{\sin^2\pi(x-y)}\right\} dx\,dy$$

for any C^2-embedding $\varphi: S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ such that $|\varphi'| = 1$. This quantity is admitted to be negative, but it is bounded from below ($E_{\text{knot}}(\varphi) \geq -2$) and blows up if a knot has a self-intersection. O'Hara showed that there exist only finitely many ambient isotopy classes of knots whose energy and bending energy (see [4] for the definition) are bounded from above by any given constant.

In his prize winning paper [3] for the Plateau problem, Douglas employed the *Douglas functional*

$$\begin{aligned} D(\varphi) &= \frac{1}{16\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{|\hat{\varphi}(e^{\sqrt{-1}\alpha}) - \hat{\varphi}(e^{\sqrt{-1}\beta})|^2}{\sin^2(\alpha-\beta/2)}\,d\alpha\,d\beta \\ &= \frac{\pi}{4}\int\int_{S^1\times S^1}\frac{|\varphi(x)-\varphi(y)|^2}{\sin^2\pi(x-y)}\,dx\,dy, \end{aligned}$$

where $\hat{\varphi}(e^{\sqrt{-1}\alpha}) = \varphi(\alpha/2\pi)$. (See also Struwe [7].) The Douglas functional is closely related to the *Dirichlet integral* (or more generally the *energy functional*) of a C^1-map $f: B^2 \to \mathbb{R}^3$

$$E(f) = \frac{1}{2}\int\int_{B^2}\|df\| = \frac{1}{2}\int\int_{B^2}\left(\left\|\frac{\partial f}{\partial x}\right\|^2 + \left\|\frac{\partial f}{\partial y}\right\|^2\right) dx\,dy,$$

where $B^2 = \{z = x + \sqrt{-1}y \in \mathbb{C};\ |z| < 1\}$. Indeed $D(\varphi)$ is equal to the energy $E(f_\varphi)$ of the harmonic extension f_φ of φ, i.e., $\Delta f_\varphi = 0$ in B^2, $f_\varphi = \hat{\varphi}$ on ∂B^2. (See Douglas [3], Ahlfors [1, Theorems 2–5].) Courant [2] minimized the Dirichlet functional to solve the Plateau problem. The Douglas functional of a curve φ is an energy of the harmonic surface f_φ with the boundary φ. In other

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words, the Douglas functional is a "tension" of a strained surface spanning the curve φ. In contrast, O'Hara's energy of a knotted curve presents a "repulsion" between segments of the curve. If the curve shrinks, O'Hara's energy increases while the Douglas functional decreases. In this note, we show the following relation between these two energies.

Theorem. $(E_{\text{knot}}(\varphi)+4)D(\varphi) \geq 1/2\pi$.

This result does not seem to be sharp, since our proof uses only the Schwarz inequality.

Proof of Theorem. Let $\lambda\colon S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be a function defined by $\lambda(x) = \min\{x, 1-x\}$ $(x \in [0, 1])$. Note that λ can be regarded as a piecewise linear periodic function on $\mathbb{R}$, which has the same order at the corners as the function $|x|$. Since $\sin \pi x \leq \pi x$ $(0 \leq x \leq \frac{1}{2})$, we see

$$|\sin \pi x| \leq \pi\lambda(x). \tag{1}$$

We claim

$$|\varphi(x) - \varphi(y)| \leq \lambda(|x-y|). \tag{2}$$

Indeed if $|x-y| \leq \frac{1}{2}$, we know $|\varphi(x)-\varphi(y)| \leq |x-y| = \lambda(|x-y|)$, since $|\varphi'| = 1$. Suppose $|x-y| > \frac{1}{2}$. Without loss of generality, we may assume $0 \leq x < y < 1$. Then $y > x + \frac{1}{2}$; hence, $\lambda(|x-y|) = 1-(y-x)$. Since φ is periodic on $\mathbb{R}$, we have $|\varphi(x)-\varphi(y)| = |\varphi(x+1)-\varphi(y)| \leq |x+1-y| = \lambda(|x-y|)$.

We see that

$$\begin{aligned}
&\int\!\!\int_{S^1\times S^1} \left(\frac{\pi^2}{\sin^2 \pi(x-y)} - \frac{1}{\lambda(|x-y|)^2}\right) dx\,dy \\
&= \int_0^1 \int_y^{y+1/2} \left(\frac{\pi^2}{\sin^2 \pi(x-y)} - \frac{1}{|x-y|^2}\right) dx\,dy \\
&\quad + \int_0^1 \int_{y+1/2}^{y+1} \left(\frac{\pi^2}{\sin^2 \pi(x-y)} - \frac{1}{(1-|x-y|)^2}\right) dx\,dy \\
&= \int_0^1 \lim_{\varepsilon\downarrow 0} \int_{y+\varepsilon}^{y+1/2} \left(\frac{\pi^2}{\sin^2 \pi(x-y)} - \frac{1}{|x-y|^2}\right) dx\,dy \\
&\quad + \int_0^1 \lim_{\varepsilon\downarrow 0} \int_{y+1/2}^{y+1-\varepsilon} \left(\frac{\pi^2}{\sin^2 \pi(x-y)} - \frac{1}{(1-|x-y|)^2}\right) dx\,dy \\
&= \int_0^1 \lim_{\varepsilon\downarrow 0} \int_{\varepsilon}^{1/2} \left(\frac{\pi^2}{\sin^2 \pi r} - \frac{1}{r^2}\right) dr\,dy \\
&\quad + \int_0^1 \lim_{\varepsilon\downarrow 0} \int_{1/2}^{1-\varepsilon} \left(\frac{\pi^2}{\sin^2 \pi r} - \frac{1}{(1-r)^2}\right) dr\,dy \\
&= 2\int_0^1 \lim_{\varepsilon\downarrow 0} \int_{\varepsilon}^{1/2} \left(\frac{\pi^2}{\sin^2 \pi r} - \frac{1}{r^2}\right) dr\,dy \\
&= 2\int_0^1 \lim_{\varepsilon\downarrow 0} \left\{\pi \cot \pi\varepsilon - \frac{1}{\varepsilon} + 2\right\} dy = 4.
\end{aligned}$$

Then we have

$$\begin{aligned}
E_{\text{knot}}(\varphi) &= \frac{1}{2}\int_0^1\int_0^1 \left\{ \frac{1}{|\varphi(x)-\varphi(y)|^2} - \frac{1}{\lambda(|x-y|)^2} \right\} dx\,dy \\
&\quad - \frac{1}{2}\int_0^1\int_0^1 \left\{ \frac{\pi^2}{\sin^2 \pi(x-y)} - \frac{1}{\lambda(|x-y|)^2} \right\} dx\,dy \\
&= \frac{1}{2}\int_0^1\int_0^1 \left\{ \frac{1}{|\varphi(x)-\varphi(y)|^2} - \frac{1}{\lambda(|x-y|)^2} \right\} dx\,dy - 2 \\
&= \frac{1}{2}\int_0^1\int_0^1 \frac{1}{\lambda(|x-y|)^2}\left\{ \frac{\lambda(|x-y|)^2}{|\varphi(x)-\varphi(y)|^2} - 1 \right\} dx\,dy - 2 \\
&\geq 2\int_0^1\int_0^1 \left\{ \frac{\lambda(|x-y|)^2}{|\varphi(x)-\varphi(y)|^2} - 1 \right\} dx\,dy - 2 \quad \left(\because\ (2) \text{ and } \lambda \leq \frac{1}{2}\right) \\
&= 2\int_0^1\int_0^1 \frac{\lambda(|x-y|)^2}{|\varphi(x)-\varphi(y)|^2}\, dx\,dy - 4,
\end{aligned}$$

i.e.,

$$E_{\text{knot}}(\varphi) + 4 \geq 2\int_0^1\int_0^1 \frac{\lambda(|x-y|)^2}{|\varphi(x)-\varphi(y)|^2}\, dx\,dy.$$

On the other hand, (1) implies that

$$D(\varphi) \geq \frac{1}{4\pi}\int_0^1\int_0^1 \frac{|\varphi(x)-\varphi(y)|^2}{\lambda(|x-y|)^2}\, dx\,dy.$$

Then, by the Schwarz inequality, we have

$$\begin{aligned}
&(E_{\text{knot}}(\varphi) + 4)D(\varphi) \\
&\quad \geq 2\int_0^1\int_0^1 \frac{\lambda(|x-y|)^2}{|\varphi(x)-\varphi(y)|^2}\, dx\,dy \frac{1}{4\pi}\int_0^1\int_0^1 \frac{|\varphi(x)-\varphi(y)|^2}{\lambda(|x-y|)^2}\, dx\,dy \\
&\quad \geq \frac{1}{2\pi}\left\{ \int_0^1\int_0^1 dx\,dy \right\}^2 = \frac{1}{2\pi}. \quad \square
\end{aligned}$$

Remark. Let φ_0 be a standard circle, i.e., $\varphi_0(x) = (\frac{1}{2\pi}\cos 2\pi x\,,\ \frac{1}{2\pi}\sin 2\pi x\,,\ 0)$. We see

$$\begin{aligned}
|\varphi_0(x) - \varphi_0(y)|^2 &= \frac{1}{4\pi^2}\{(\cos 2\pi x - \cos 2\pi y)^2 + (\sin 2\pi x - \sin 2\pi y)^2\} \\
&= \frac{1}{4\pi^2}\{2 - 2(\cos 2\pi x \cos 2\pi y + \sin 2\pi x \sin 2\pi y)\} \\
&= \frac{1}{2\pi^2}\{1 - \cos 2\pi(x-y)\} = \frac{1}{\pi^2}\sin^2 \pi(x-y);
\end{aligned}$$

hence, we know $E_{\text{knot}}(\varphi_0) = 0$ and $D(\varphi_0) = \frac{1}{4\pi}$. Then $(E_{\text{knot}}(\varphi_0) + 4)D(\varphi_0) = \frac{1}{\pi}$.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, YAMAGUCHI UNIVERSITY, YAMAGUCHI 753, JAPAN
E-mail address: e76220@sinet.ad.jp