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ON IMMERSIONS OF k-CONNECTED n-MANIFOLDS

B. H. LI AND M. E. MAHOWALD

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ABSTRACT. In this note we classify up to regular homotopy the classes of immersions of k-connected closed differentiable n-manifolds in $\mathbb{R}^{2n-k}$.

In [W] it was claimed that for a k-connected closed differentiable manifold M of dimension n with $0 \leq 2k \leq n-2$, the regular homotopy classes of M in $\mathbb{R}^{2n-k}$ are in one to one correspondence with $\pi_n(V_{2n-k,n})$, where $V_{2n-k,n}$ is a Stiefel manifold. But this result is incorrect. In this note, we will present the correct answer and give counterexamples to the claim of [W].

To state our result, some definitions are needed. First, let $\pi_i(M)$ be the ith homotopy group of M. Then

$$h : \pi_{k+1}(M) \to \widetilde{KO}(S^{k+1})$$

is defined as follows: If $\alpha \in \pi_{k+1}(M)$ is represented by a map $\tilde{\alpha} : S^{k+1} \to M$, then $h(\alpha)$ is the element in $\widetilde{KO}(S^{k+1})$ represented by $2\tilde{\alpha}^*\nu_M$, where ν_M is the stable normal bundle of M. It is easy to see that h is well defined and is a homomorphism.

Next, let

$$k : \widetilde{KO}(S^{k+1}) \to \pi_k(SO)$$

be the natural isomorphism,

$$J : \pi_k(SO) \to \pi_k^s = \pi_n(S^{n-k})$$

be the J-homomorphism, and let

$$S^{n-k} \xrightarrow{i} V_{2n-k,n} \to V_{2n-k,n-1}$$

be the natural fibration. Let

$$e = i_* \circ J \circ k \circ h \ : \ \pi_{k+1}(M) \to \pi_n(V_{2n-k,n}).$$

Then we have

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Theorem. *If* $0 \leq 2k \leq n-2$ *and* M *is a* k*-connected closed differentiable manifold, then the regular homotopy classes of immersions of* M *in* $\mathbb{R}^{2n-k}$ *are in one-to-one correspondence with the cokernel of* e.

Example. Let $\mathbb{K}^2$ be the Cayley projective plane and $M = \mathbb{K}^2 \times S^{23}$. Then M is 7-connected with dimension 39, and the homotopy sequence of the fibration $S^{32} \to V_{71,39} \to V_{71,38}$ gives the split exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_{39}(S^{32}) & \xrightarrow{i_*} & \pi_{39}(V_{71,39}) & \longrightarrow & \pi_{39}(V_{71,38}) & \longrightarrow & 0 \\ & & \downarrow\cong & & & & \downarrow\cong & & \\ & & \mathbb{Z}_{240} & & & & \mathbb{Z}_{12} \oplus \mathbb{Z}_{16} & & \end{array}$$

Now, the tangent bundle of $\mathbb{K}^2$ restricted to $\mathbb{K}^1 \cong S^8$ is stably equivalent to the canonical 8-dimensional vector bundle, which can be defined by regarding $S^8 \cong \mathbb{R}^8 \cup \{\infty\} = D_+ \cup D_-$ and $\mathbb{R}^8$ as the Cayley algebra. Let $D_+ = \{x \in \mathbb{R}^8/\|x\| \leq 1\}$ and $D_- = S^8 - D_+$. Glue $D_+ \times \mathbb{R}^8$ and $D_- \times \mathbb{R}^8$ over $D_+ \cap D_- = S^7$ by $(x, y) \sim (x, xy)$ where xy is the product of $x \in S^7$ and $y \in \mathbb{R}^8$ as Cayley numbers (see Steenrod [S, p. 109]).

Let $1, e_1, \ldots, e_7$ be the units of Cayley algebra. Then $x \to (x, xe_1, \ldots, xe_7)$ defines a map $S^7 \to SO(8)$, which represents a generator of the first summand of $\pi_7(SO(8)) \cong \pi_7(S^7) \oplus \pi_7(SO(7)) \cong \mathbb{Z} \oplus \mathbb{Z}$, hence a generator of $\pi_7(SO)$ (cf. [L, Remark 4]). Thus $2T(M)|_{\mathbb{K}^1}$ represents twice a generator of $\widetilde{KO}(S^8)$. Since $J = \pi_7(SO) \to \pi_7^S$ is surjective, we see that the cokernel of e is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{16}$. This shows that the result of [W] is incorrect.

We can produce more examples. Let ξ be any 8-dimensional vector bundle over S^8. Then the second Pontriagin class $p_2(\xi) = 6q$ for some $q \in \mathbb{Z}$ (cf. [K]). Let $M = S(\xi \oplus 1) \times S^{23}$, where $S(\xi \oplus 1)$ is the total space of the sphere bundle of $(\xi \oplus 1)$. Then $\pi_8(M) = Z \oplus Z$ and h maps one summand to 0, another to $2q\mathbb{Z}$. Thus the image of e can be any subgroup of $\pi_7^S \cong \mathbb{Z}_{240}$ consisting of even elements, and only when $q \equiv 0 \bmod 120$ is the result of [W] true.

Since $2\widetilde{KO}(S^{k+1}) = 0$, if $k \not\equiv 3, 7 \bmod 8$, we have

Corollary 1. *If* $k \not\equiv 3, 7 \bmod 8$, *then the cokernel of* e *is* $\pi(V_{2n-k,n})$.

If $n-k$ is odd, then $i_* = \pi_n(S^{n-k}) \to \pi_n(V_{2n-k,n})$ sends $2\pi_n(S^{n-k})$ to zero since the first two essential cells of the Stiefel manifold form a $Z/2$ Moore space. So we have

Corollary 2. *If* $n-k$ *is odd and* $k \not\equiv 3, 7 \bmod 8$, *then the image of* e *is zero.*

Proof of the theorem. By using normal bordism theory, we see from [Ko] or [D] that $\mathrm{Imm}[M, \mathbb{R}^{2n-k}]$ (or $[M \propto \mathbb{R}^{2n-k}]$) $\cong \Omega_k(M \times P^\infty, \phi)$, where P^∞ is the infinite real projective space, $\phi = (2n-k)\lambda - \lambda \otimes T(M) - T(M)$, and λ is the canonical line bundle over P^∞. There is an exact sequence

$$\begin{aligned} &\Omega_{k+1}(M \times P^\infty, * \times P^\infty, \phi) \xrightarrow{\partial} \Omega_k(P^\infty, \phi|_{P^\infty}) \\ &\quad \to \Omega_k(M \times P^\infty, \phi) \to \Omega_k(M \times P^\infty, * \times P^\infty, \phi) \end{aligned}$$

(cf. [D, p. 310]), where $* \in M$ is a point. Since $(M \times P^\infty, P^\infty)$ is k-connected, it follows from Proposition 5.1 in [D] that $\Omega_k(M \times P^\infty, * \times P^\infty, \phi) = 0$, and the natural homomorphism

$$\mu : \Omega_{k+1}(M \times P^{\infty}, * \times P^{\infty}, \phi) \to H_{k+1}(M \times P^{\infty}, * \times P^{\infty}, Z(\phi))$$

is an isomorphism.

Now look at the exact sequence

$$\begin{aligned} \to H_{k+1}(P^{\infty}, 2(\phi|_{P^{\infty}})) &\xrightarrow{i_{k+1}} H_{k+1}(M \times P^{\infty}, z(\phi)) \\ &\xrightarrow{j} H_{k+1}(M \times P^{\infty}, * \times P^{\infty}, z(\phi)) \\ &\xrightarrow{\partial} H_k(P^{\infty}, z(\phi|_{P^{\infty}})) \xrightarrow{i_k} H_k(M \times P^{\infty}, z(\phi)). \end{aligned}$$

Since $W_1(\phi) = W_1((n-k)\lambda)$, we have $Z(\phi) = Z \otimes Z((n-k)\lambda)$. Hence, by the Künneth formula for twisted coefficients,

$$\begin{aligned} &H_*(M \times P^{\infty}, Z(\phi)) \\ &\quad = H_*(P^{\infty}, Z((n-k)\lambda)) \oplus H_*(M, Z) \otimes H_0(P^{\infty}, Z((n-k)\lambda)) \end{aligned}$$

for $* \leq k+1$. This shows that i_k is an isomorphism and j is an isomorphism from $H_{k+1}(M, Z) \otimes H_0(P^{\infty}, \mathbb{Z}((n-k)\lambda))$ to $H_{k+1}(M \times P^{\infty}, P^{\infty}, Z(\phi))$.

Now $\Omega_k(P^{\infty}, \phi|_{P^{\infty}}) = \Omega_k(P^{\infty}, (n-k)\lambda) = \pi_n(V_{2n-k,n})$ (cf. [D, Proposition 7.3] or [Ko, Proposition 5.4]). We need only to calculate the image of ∂ in the following diagrams:

$$\begin{array}{ccc} \Omega_{k+1}(M \times P^{\infty}, P^{\infty}, \phi) & \xrightarrow{\partial} & \Omega_k(P^{\infty}, \phi|_{P^{\infty}}) \\ \downarrow \cong & & \downarrow \cong \\ H_{k+1}(M, Z) \otimes H_0(P^{\infty}, z((n-k)\lambda)) & & \pi_n(V_{2n-k,n}) \end{array}$$

By the Hurewicz isomorphism theorem,

$$H_{k+1}(M) \cong H_{k+1}(M, \mathbb{Z}) \quad \text{if } k > 0,$$

$$\frac{\pi_1(M)}{[\pi_1(M), \pi_1(M)]} \cong H_1(M, \mathbb{Z}) \quad \text{if } k = 0.$$

Let $S^{k+1} = D_+ \cup D_-$, where D_+ and D_- are disks with $D_+ \cap D_- = S^k$, and $c : S^{k+1} \to P^{\infty}$ be a constant map. For any $\alpha \in \pi_{k+1}(M, *)$, we can choose $\tilde{\alpha} : S^{k+1} \to M$, representing α with $\tilde{\alpha}(D_-) = *$.

Let $\bar{\alpha} = (\tilde{\alpha}, c) : D_+ \to M \times P^{\infty}$. Then $\bar{\alpha}$ maps $(D_+, \partial D_+)$ into $(M \times P^{\infty}, * \times P^{\infty})$. Regard ϕ as a stable bundle; then $T(D_+) \oplus \bar{\alpha}^* \phi$ obviously has a trivialization V. Thus the triple $((D_+, \partial D_+), \bar{\alpha}, V)$ defines an element in $\Omega_{k+1}(M \times P^{\infty}, * \times P^{\infty}, \phi)$, and $(\partial D_+, \bar{\alpha}|_{\partial D_+}, V|_{\partial D_+})$ defines an element in $\Omega_k(*, \text{trivial}) \equiv \pi_k^s$, which can be regarded as an element of $\Omega_k(P^{\infty}, \phi|_{P^{\infty}}$ via the inclusion $* = c(S^{k+1}) \subset P^{\infty}$. From the isomorphisms

$$\begin{aligned} \Omega_{k+1}(M \times P^{\infty}, * \times P^{\infty}, \phi) &\cong H_{k+1}(M \times P^{\infty}, * \times P^{\infty}, Z(\phi)) \\ &\cong H_{k+1}(M, Z(\phi)) \otimes H_0(P^{\infty}, Z((n-k)\lambda)) \\ &\cong \pi_{k+1}(M) \otimes H_0(P^{\infty}, Z((n-k)\lambda). \end{aligned}$$

We see that the image of

$$\partial : \Omega_{k+1}(M \times P^{\infty}, * \times P^{\infty}, \phi) \to \Omega_k(P^{\infty}, \phi|_{P^{\infty}})$$

is exactly the set $\{[(\partial D_+, \overline{\alpha}|_{\partial D_+}, V|_{\partial D_+})]/\alpha \in \pi_{k+1}(M)\}$.

Regard $V|_{\partial D_+}$ as a trivialization of

$$(T(D_-) \oplus \tilde{\alpha}^*\phi)|_{\partial D_-};$$

then it defines an element of $\pi_k(SO)$, which is the value of $k : \widetilde{KO}(S^{k+1}) \to \pi_k(SO)$ on the stable bundle $\tilde{\alpha}^*(2\nu_M) = h(\alpha)$. It is obvious that the element in $\Omega_k(*, \text{trivial})$ defined by $(\partial D_+, \tilde{\alpha}|_{\partial D_+}, V|_{\partial D_+})$ is $(J \circ k \circ h)(\alpha)$.

Now, the only remaining thing is to see that the diagram

$$\begin{array}{ccc} \Omega_k(*, \text{trivial}) & \longrightarrow & \Omega_k(P^\infty, (n-k)\lambda) \\ \downarrow\cong & & \downarrow\cong \\ \pi_n(S^{n-k}) & \xrightarrow{i_*} & \pi_n(V_{2n-k,\ n}) \end{array}$$

commutes. First

$$\Omega_k(P^\infty, (n-k)\lambda) \cong \Omega_k(P^{m-1}, (n-k)\lambda) \cong \pi_n\left(\frac{P^{2n-k-1}}{P^{n-k-1}}\right)$$

where P^{2n-k-1}/P^{n-k-1} is the Thom space of $(n-k)\lambda$. Under the natural map

$$P^{2n-k-1}/P^{n-k-1} \to V_{2n-k,n}$$

the Thom space of $(n-k)\lambda$ restricted to a point maps onto $i(S^{n-k}) \subset V_{2n-k,n}$. So the above diagram commutes. So far we have proved that in the following diagram

$$\begin{array}{ccccc} \Omega_{k+1}(M \times P^\infty, * \times P^\infty, \phi) & \xrightarrow{\partial} & \Omega_k(P^\infty, \phi|_{P^\infty}) \to \Omega_k(M \times P^\infty, \phi) \to 0 \\ & & \downarrow\cong \\ \pi_{k+1}(M) & \xrightarrow{e=i_*\circ J\circ k\circ h} & \pi_n(V_{2n-k,n}) \end{array}$$

the images of ∂ and e are isomorphic and hence so is the theorem.

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Insitute of Systems Science, Academia Sinica, Beijing 100080, People's Republic of China

Department of Mathematics, Northwestern University, Evanston, Illinois 60201
E-mail address: mark@math.nwu.edu