PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 118, Number 1, May 1993

INFINITE-ODS IN ARCWISE CONNECTED CONTINUA

ELDON J. VOUGHT

(Communicated by James E. West)

ABSTRACT. The theorem is proved that a continuum that has only finitely many arc components (hence an arcwise connected continuum) and contains an n-od for every integer n must contain an infinite-od.

FitzGerald [1, Corollary 4.8, p. 157] showed that a locally connected continuum that is an n-od for every positive integer n is an ∞-od. Heath obtained the same conclusion by replacing locally connected with a hereditary decomposable condition (see Theorem A). In the same paper [2, Example 2, p. 477] she constructed a continuum that is an n-od for every positive integer n but is not an ∞-od. Nall in [3, p. 245], generalizing a result of Sorgenfrey, gave sufficient conditions in order for a continuum to contain an ∞-od. In a conversation with the author, Nall and Hagopian raised the question of whether an arcwise connected continuum that contains an n-od for every n must contain an ∞-od. The purpose of this paper is to give an affirmative answer to this question. In fact, the answer is yes under the weaker hypothesis that the continuum has only finitely many arc components. In the case of infinitely many arc components, however, the answer is no, and a continuum is constructed with an infinite number of arc components that contains an n-od for every n but no ∞-od.

A continuum is a *θ_n-continuum* if no subcontinuum separates it into more than n components. A continuum X is an *n-od* if there exists a subcontinuum H, called the *hub*, such that $X\backslash H$ has at least n components. If $X\backslash H$ has an infinite number of components then X is an *∞-od*. Note that a continuum X is a θ_n-continuum if and only if X is not an $(n+1)$-od.

Example. There exists a continuum with an infinite number of arc components that contains an n-od for every n but does not contain an ∞-od.

Let X be a continuum irreducible between a and b that admits a monotone upper semicontinuous map f onto $[0, 1]$ such that (i) $f^{-1}(0) = a$, $f^{-1}(1) = b$, (ii) $f^{-1}(y)$ is degenerate if $y \neq \frac{1}{n}$ for $n = 3, 4, \ldots$, (iii) $f^{-1}(\frac{1}{n})$ is a simple n-od of diameter $\frac{1}{n}$ for $n = 3, 4, \ldots$, and (iv) $f^{-1}(\frac{1}{n})$

Received by the editors September 3, 1991; presented to the Annual Spring Topology Conference at California State University, Sacramento on April 11, 1991.

1991 *Mathematics Subject Classification*. Primary 54F20; Secondary 54B15.

Key words and phrases. Infinite-od (∞-od), n-od, arc component, θ_n-continuum, hereditarily decomposable, monotone upper semicontinuous decomposition, graph.

has void interior for $n = 3, 4, \ldots$. The continuum X has the aforementioned properties.

To proceed, we need the following two theorems.

Theorem A (Heath). *If W is a continuum and, for each n there is a subcontinuum K_n of W such that* (i) *$W \backslash K_n$ has at least n components and* (ii) *the closure of each component of $W \backslash K_n$ is hereditarily decomposable, then W is an ∞-od* [2, *Theorem* 2, *p.* 486].

Theorem B (Vought). *If W is a hereditarily decomposable, θ_n-continuum, then W admits a unique monotone, upper semicontinuous decomposition $\mathscr{D}$, the elements of which have void interior and for which $W/\mathscr{D}$ is a graph* [4, *Theorem* 2, *p.* 635].

Lemma 1. *Let X be a continuum with only finitely many arc components and suppose X contains no ∞-od. Then every subcontinuum of X has only finitely many arc components and thus is decomposable. Hence X is hereditarily decomposable.*

Proof. Suppose W, a subcontinuum of X, has an infinite number of arc components. Since X has only finitely many arc components, let $C_1, C_2, \ldots$ be a sequence of arc components of W that lie in the same arc component C of X. Because C is arcwise connected, $C \backslash \bigcup_{i=1}^{\infty} C_i \neq \varnothing$. Choose x in $C \backslash \bigcup_{i=1}^{\infty} C_i$, w in C_1, and let $A_1 = \langle w, x \rangle$ be an arc in C from w to x. Let $w' = \mathrm{lub}_{A_1}\{y | y \in C_1\}$. Then $w' \in C_1$. For, suppose not. There exists a sequence of points of $A_1 \cap C_1$, $w_1, w_2, \ldots$, such that $w_i < w'$ for $i = 1, 2, \ldots$ and $\lim w_i = w'$. The subarc $\langle w_1, w' \rangle$ of A_1 is not contained in W, so there is a subarc $\langle r_1, s_1 \rangle$ of $\langle w_1, w' \rangle$ such that $r_1, s_1 \in W$ and $(r_1, s_1) \cap W = \varnothing$. Without loss of generality, assume that $w_2 \geq s_1$. Then there exists a subarc $\langle r_2, s_2 \rangle$ of the subarc $\langle w_2, w' \rangle$ of A_1 such that $r_2, s_2 \in W$ and $(r_2, s_2) \cap W = \varnothing$. Continuing, we obtain an infinite sequence $\langle r_1, s_1 \rangle, \langle r_2, s_2 \rangle, \ldots$ of subarcs of A_1 such that for $i = 1, 2, \ldots, r_i, s_i \in W$, $(r_i, s_i) \cap W = \varnothing$, and $(r_i, s_i) \cap (r_j, s_j) = \varnothing$ if $i \neq j$. Then $W \cup \bigcup_{i=1}^{\infty} \langle r_i, s_i \rangle$ is an ∞-od with hub W, a contradiction. So $w' \in A_1 \cap C_1$.

There exists a point $v > w'$ in A_1 such that the arc $\langle w', v \rangle \cap W = \{w'\}$. For, suppose not. There exists a sequence of points $v_1, v_2, \ldots$ in A_1 such that, for $i = 1, 2, \ldots, w' < v_i$, $v_i \in W$ and $\lim v_i = w'$. But $w' \in C_1$ and $w' = \mathrm{lub}_{A_1}\{y | y \in C_1\}$, so $\langle w', v_i \rangle \not\subset W$ for $i = 1, 2, \ldots$. Then by an analogous construction to that of the last paragraph, an ∞-od can be constructed that yields a contradiction. Let K_1 be a subarc of $\langle w', v \rangle$ with one end point w' such that $\mathrm{diam}\, K_1 \leq 1$.

Repeat the above argument with the arc component C_2 and obtain an arc K_2 such that K_2 intersects W only at an end point of K_2 and $\mathrm{diam}\, K_2 \leq \frac{1}{2}$. Clearly K_2 can be obtained such that $K_1 \cap K_2 = \varnothing$. Then in general, there is a sequence of mutually disjoint arcs $K_1, K_2, \ldots$ such that for each i, $i = 1, 2, \ldots, K_i$ intersects W only at an end point of K_i and $\mathrm{diam}\, K_i \leq 1/i$. Thus $W \cup \bigcup_{i=1}^{\infty} K_i$ is an ∞-od with hub W, a contradiction. This proves Lemma 1.

Lemma 2. *Let X be a continuum with only finitely many arc components and suppose X contains no ∞-od. Then every subcontinuum of X is a θ_n-continuum for some n.*

Proof. Let W be a subcontinuum of X and suppose W is an n-od for every n. Thus Theorem A(i) is true. By Lemma 1, W is hereditarily decomposable and so Theorem A(ii) is true. Hence W is an ∞-od, a contradiction. Therefore, there exists an integer n such that W is a θ_n-continuum and Lemma 2 is established.

Lemma 3. *Let X be a continuum with only finitely many arc components and suppose X contains no ∞-od. Then every subcontinuum admits a unique monotone, upper semicontinuous decomposition, the elements of which have void interior and for which the quotient space is a graph. Furthermore, at most a finite number of the elements of the decomposition are nondegenerate.*

Proof. Let W be a subcontinuum of X. By Lemma 1, W is hereditarily decomposable. By Lemma 2, W is a θ_n-continuum for some n. Thus, by Theorem B, W admits the decomposition $\mathscr{D}$ of Lemma 3 for which $W/\mathscr{D}$ is a graph. By Lemma 1, W has only finitely many arc components. Since $W/\mathscr{D}$ is a graph, $\mathscr{D}$ contains only finitely many elements of order different from 2, where the order of an element of $\mathscr{D}$ is its order as an element of $W/\mathscr{D}$. There can be only finitely many nondegenerate elements of $\mathscr{D}$ of order 2; for otherwise, there would be infinitely many of these elements that are points on one edge of the graph $W/\mathscr{D}$, thus contradicting the fact that W has only finitely many arc components.

Theorem (Main result). *If X is a continuum with only finitely many arc components and if X contains an n-od for every n, then X contains an ∞-od.*

Proof. Suppose X contains no ∞-od and let $X_0 = X$. By Lemma 3, X_0 has a decomposition $\mathscr{D}_0$ such that $X_0/\mathscr{D}_0$ is a graph and at most a finite number of the elements of $\mathscr{D}_0$ are nondegenerate. Since $X_0/\mathscr{D}_0$ is a graph, there is a maximum integer n_0 such that $X_0/\mathscr{D}_0$ contains an n_0-od. But X_0 contains an n-od for every n, so it follows that an element X_1 of $\mathscr{D}_0$ has the property that X_1 contains an n-od for every n. By Lemma 3 again, X_1 has a decomposition $\mathscr{D}_1$ to a graph and at most a finite number of the elements of $\mathscr{D}_1$ are nondegenerate. Since $X_1/\mathscr{D}_1$ is a graph, there is a maximum integer n_1 such that $X_1/\mathscr{D}_1$ contains an n_1-od. But X_1 contains an n-od for every n, so it follows that an element X_2 of $\mathscr{D}_1$ has the property that X_2 contains an n-od for every n.

Continuing, we obtain continua $X_0, X_1, X_2, \ldots$ and decompositions $\mathscr{D}_0$, $\mathscr{D}_1, \mathscr{D}_2, \ldots$ such that for $k = 1, 2, \ldots, X_k$ is a nondegenerate element of the decomposition $\mathscr{D}_{k-1}$ of X_{k-1} to a graph and X_k contains an n-od for every n. Since X_k is a nondegenerate element of $\mathscr{D}_{k-1}$ and X_{k-1} has only finitely many arc components, for each k, $k = 1, 2, \ldots,$ there is a half ray H_{k-1} in X_{k-1} that limits on a nondegenerate subcontinuum of X_k and intersects no nondegenerate or vertex element of $\mathscr{D}_{k-1}$. The collection of these half rays is infinite but X contains only finitely many arc components, so an infinite number of half rays lie in the same arc component of X. Without loss of generality, assume that for each k, $k = 0, 1, \ldots, H_k$ lies in the same arc component of X. Pick x_0 in H_0. Since $X_k/\mathscr{D}_k$ is a graph for each k, $k = 1, 2, \ldots$, there is a half ray R_k in H_k with end point x_k and with the same limiting set as H_k that intersects none of the at most finite number of arcs from x_0 to X_k. For each k, $k = 1, 2, \ldots,$ let A_k be an arc from x_0 to x_k

and let C_k be an arc in R_k such that $A_k \cap C_k = \{x_k\}$. Let $H = \overline{\bigcup_{i=1}^{\infty} A_i}$ and $K = H \cup (\overline{\bigcup_{i=1}^{\infty} C_i})$. Observe that $(\overline{\bigcup_{i=1}^{\infty} A_i}) \backslash (\bigcup_{i=1}^{\infty} A_i)$ is a subset of $\bigcap_{i=1}^{\infty} X_i$. This follows from the fact that for every n, $X_n/\mathscr{D}_n$ is a graph, and so there are only a finite number of arcs from x_0 to X_{n+1}. Because the X_i's are nested, it is true that $(\overline{\bigcup_{i=1}^{\infty} C_i}) \backslash (\bigcup_{i=1}^{\infty} C_i)$ is also a subset of $\bigcap_{i=1}^{\infty} X_i$. Thus K is an ∞-od with hub H. This is a contradiction and the theorem is established.

An immediate application of the theorem yields the following

Corollary. *If X is an arcwise connected continuum that contains an n-od for every n, then X contains an ∞-od.*

References

1. R. W. FitzGerald, *Connected sets with a finite disconnection property*, Academic Press, New York, 1975, pp. 139–173.
2. Jo Heath, *On n-ods*, Houston J. Math. **9** (1983), 477–487.
3. Van C. Nall, *On the presence of n-ods and infinite-ods*, Houston J. Math. **15** (1989), 245–247.
4. Eldon J. Vought, *ω-connected continua and Jones' K function*, Proc. Amer. Math. Soc. **91** (1984), 633–636.

Department of Mathematics and Statistics, California State University, Chico, California 95929-0001

E-mail address: Vought@ecst.csuchico.edu