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ON THE DIOPHANTINE EQUATIONS $d_1x^2 + 2^{2m}d_2 = y^n$ AND $d_1x^2 + d_2 = 4y^n$

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ABSTRACT. Let d_1, d_2 be coprime positive integers, which are squarefree, and let h denote the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d_1d_2})$. Let m, n be integers such that $m \geq 0$, $n > 1$, and $\gcd(n, 2h) = 1$. In this paper we prove that if $n \geq 8.5 \cdot 10^6$, then the equations $d_1x^2 + 2^{2m}d_2 = y^n$ $(2 \nmid y)$ and $d_1x^2 + d_2 = 4y^n$ have no positive integer solutions (x, y) with $\gcd(x, y) = 1$.

Let d_1, d_2 be coprime positive integers, which are squarefree, and let h denote the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d_1d_2})$. Let m, n be integers such that $m \geq 0$, $n > 1$, and $\gcd(n, 2h) = 1$. There have been many papers concerned with the solvability of the diophantine equations:

$$(1) \qquad d_1x^2 + 2^{2m}d_2 = y^n, \qquad x > 0,\ y > 1,\ 2 \nmid y,\ \gcd(x, y) = 1,$$

and

$$(2) \qquad d_1x^2 + d_2 = 4y^n, \qquad x > 0,\ y > 1,\ \gcd(x, y) = 1.$$

The known results include the following:

1 (Blass [1]). If $d_1 = 1$, $d_2 \neq 19$ or 341, $m = 0$, and $n = 5$, then (1) has no integer solution (x, y).

2 (Blass and Steiner [2]). If $d_1 = 1$, $m = 0$, and $n = 7$, then (1) has no integer solution (x, y).

3 (Nagell [16, 17], Ljunggren [9, 12], Cardell [3]). If $d_1 = 1$, $m \leq 1$, and d_2 satisfies some congruence conditions, then (1) has no integer solution (x, y).

4 (Ljunggren [10–12], Krohonen [4–7]). If $\min(d_1, d_2) > 1$, $m \leq 1$, and d_1, d_2 satisfy some congruence conditions then (1) has no integer solution (x, y).

5 (Persson [18], Stolt [19]). If $d_1 = 1$ and n is a fixed odd prime, then there exist only a finite number of d_2 for which (2) has integer solutions (x, y) and the number of solutions is finite.

6 (Ljunggren [13, 14]). If $d_1 = 1$ and d_2 satisfies some congruence conditions, then there exist only a finite number of n for which (2) has integer solutions (x, y).

In this paper we prove a general result as follows:

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Theorem. *If* $n \geq 8.5 \cdot 10^6$, *then* (1) *and* (2) *have no integer solution* (x, y).

In order to prove the theorem, we now introduce a result concerned with the linear forms in logarithms, which was derived by Mignotte and Waldschmidt [15]. Let α be a nonzero algebraic number with the defining polynomial

$$a_0 z^r + a_1 z^{r-1} + \cdots + a_r = a_0(z - \sigma_1\alpha)\cdots(z - \sigma_r\alpha), \qquad a_0 > 0,$$

where $\sigma_1\alpha, \ldots, \sigma_r\alpha$ are all the conjugates of α. Then

$$h(\alpha) = \frac{1}{r}\left(\operatorname{Log} a_0 + \sum_{i=1}^{r} \operatorname{Log}\max(1, |\sigma_i\alpha|)\right)$$

is called Weil's height of α.

Lemma [15, §10]. *Let* $\log\alpha$ *be any nonzero determination of the logarithm of* α. *If* $r = 2$ *and* $\Lambda = b_1\pi\sqrt{-1}/b_2 - \log\alpha \neq 0$ *for some positive integers* b_1 *and* b_2, *then*

$$|\Lambda| > \exp(-20600A(1.35 + \operatorname{Log} B + \operatorname{Log}\operatorname{Log} 2B)^2),$$

where $A = \max(1/2, h(\alpha))$, $B = \max(b_1, b_2)$.

Proof of Theorem. Let (x, y) be an integer solution of (1). Then, according to the analysis in [10], we have

$$x\sqrt{d_1} + 2^m\sqrt{-d_2} = (a\sqrt{d_1} + b\sqrt{-d_2})^n, \tag{3}$$

where a, b are integers, which satisfy

$$d_1 a^2 + d_2 b^2 = y, \qquad \gcd(a, b) = 1. \tag{4}$$

Let $\varepsilon_1 = a\sqrt{d_1} + b\sqrt{-d_2}$, $\bar{\varepsilon}_1 = a\sqrt{d_1} - b\sqrt{-d_2}$. We get from (3) that

$$2^m = \frac{\varepsilon_1^n - \bar{\varepsilon}_1^n}{2\sqrt{-d_2}} = b\frac{\varepsilon_1^n - \bar{\varepsilon}_1^n}{\varepsilon_1 - \bar{\varepsilon}_1}. \tag{5}$$

By Waring's formula [8, Formula 1 · 76],

$$\frac{\varepsilon_1^n - \bar{\varepsilon}_1^n}{\varepsilon_1 - \bar{\varepsilon}_1} = \sum_{i=0}^{(n-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} (\varepsilon_1 - \bar{\varepsilon}_1)^{n-2i-1}(\varepsilon_1\bar{\varepsilon}_1)^i = \sum_{i=0}^{(n-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} (-4d_2b^2)^{(n-1)/2-i}y^i,$$

where

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{(n-i-1)!n}{(n-2i)!i!}, \qquad i = 0, \ldots, \frac{n-1}{2},$$

are positive integers. This implies that $(\varepsilon_1^n - \bar{\varepsilon}_1^n)/(\varepsilon_1 - \bar{\varepsilon}_1)$ is an odd integer. From (5), we get

$$\frac{\varepsilon_1^n - \bar{\varepsilon}_1^n}{\varepsilon_1 - \bar{\varepsilon}_1} = \pm 1. \tag{6}$$

Similarly, if (x, y) is an integer solution of (2), then we have

$$\frac{\varepsilon_2^n - \bar{\varepsilon}_2^n}{\varepsilon_2 - \bar{\varepsilon}_2} = \pm 1, \tag{7}$$

where

$$\varepsilon_2 = \frac{a'\sqrt{d_1} + b'\sqrt{-d_2}}{2}, \qquad \bar{\varepsilon}_2 = \frac{a'\sqrt{d_1} - b'\sqrt{-d_2}}{2},$$

and a', b' are integers, which satisfy

(8) $$d_1a'^2 + d_2b'^2 = 4y, \qquad b' = \pm 1.$$

Let

$$\varepsilon = \begin{cases} \varepsilon_1, & \text{for equation (1)}, \\ \varepsilon_2, & \text{for equation (2)}; \end{cases} \qquad \bar\varepsilon = \begin{cases} \bar\varepsilon_1 & \text{for equation (1)}, \\ \bar\varepsilon_2 & \text{for equation (2)}. \end{cases}$$

Then (6) and (7) can be written as

(9) $$|\varepsilon^n - \bar\varepsilon^n| = |\varepsilon - \bar\varepsilon|.$$

For any complex number z, we have either $|e^z - 1| > 1/2$ or $|e^z - 1| \geq |z - k\pi\sqrt{-1}|/2$ for some integers k. Hence

(10) $$\begin{aligned} \operatorname{Log}|\varepsilon^n - \bar\varepsilon^n| &= n \operatorname{Log}|\varepsilon| + \operatorname{Log}\left|\left(\frac{\bar\varepsilon}{\varepsilon}\right)^n - 1\right| \\ &\geq n \operatorname{Log}|\varepsilon| + \operatorname{Log}\left|n \log\frac{\bar\varepsilon}{\varepsilon} - k\pi\sqrt{-1}\right| - \operatorname{Log} 2, \end{aligned}$$

where k is an integer with $|k| \leq n$. By (4) and (8), $\bar\varepsilon/\varepsilon$ satisfies

(11) $$y\left(\frac{\bar\varepsilon}{\varepsilon}\right)^2 - 2(d_1a^2 - d_2b^2)\frac{\bar\varepsilon}{\varepsilon} + y = 0, \qquad \gcd(y, 2(d_1a^2 - d_2b^2)) = 1,$$

or

(12) $$y\left(\frac{\bar\varepsilon}{\varepsilon}\right)^2 - \left(\frac{d_1a'^2 - d_2b'^2}{2}\right)\frac{\bar\varepsilon}{\varepsilon} + y = 0, \qquad \gcd\left(y, \frac{d_1a'^2 - d_2b'^2}{2}\right) = 1.$$

Since $y > 1$, $\bar\varepsilon/\varepsilon$ is not a root of unity. Therefore $\Lambda = n\log(\bar\varepsilon/\varepsilon) - k\pi\sqrt{-1} \neq 0$. By (11) and (12), the degree of $\mathbb{Q}(\bar\varepsilon/\varepsilon)$ is equal to 2 and $h(\bar\varepsilon/\varepsilon) = \operatorname{Log}\sqrt{y}$. By the Lemma,

(13) $$|\Lambda| > n\exp(-20600(\operatorname{Log}\sqrt{y})(1.35 + \operatorname{Log} n + \operatorname{Log}\operatorname{Log} 2n)^2).$$

Substituting (13) into (10),

(14) $$\operatorname{Log}\frac{2}{n}|\varepsilon^n - \bar\varepsilon^n| > n\operatorname{Log}|\varepsilon| - 20600(\operatorname{Log}\sqrt{y})(1.35 + \operatorname{Log} n + \operatorname{Log}\operatorname{Log} 2n)^2.$$

Notice that $|\varepsilon| = \sqrt{y}$ and $|\varepsilon - \bar\varepsilon| < 2\sqrt{y}$. If (9) holds, then from (14) we get

$$\operatorname{Log}\frac{4}{3}\sqrt{y} + 20600(\operatorname{Log}\sqrt{y})(1.35 + \operatorname{Log} n + \operatorname{Log}\operatorname{Log} 2n)^2 > n\operatorname{Log}\sqrt{y}.$$

This is impossible for $n \geq 8.5 \cdot 10^6$. The theorem is proved.

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REFERENCES

1. J. Blass, *On the diophantine equation* $Y^2 + K = X^5$, Bull. Amer. Math. Soc. **80** (1974), 329.
2. J. Blass and R. Steiner, *On the equation* $y^2 + k = x^7$, Utilitas Math. **13** (1978), 293–297.
3. L. Cardell, *Some results on the diophantine equation* $x^2 + D = y^n$, Dep. Math., Chalmers Univ., Techn. Univ. Göteborg 1984-06, 1984.
4. O. Korhonen, *On the diophantine equation* $Ax^2 + 8B = y^n$, Acta Univ. Oulu Ser. A Sci. Rerum. Natur. Math. **16** (1979).
5. ——, *On the diophantine equation* $Ax^2 + 2B = y^n$, Acta Univ. Oulu Ser. A Sci. Rerum. Natur. Math. **17** (1979).
6. ——, *On the diophantine equation* $2Ax^2 + B = y^n$, Acta Univ. Oulu Ser. A Sci. Rerum. Natur. Math. **21** (1980).
7. ——, *On the diophantine equation* $Cx^2 + D = y^n$, Acta Univ. Oulu Ser. A Sci. Rerum. Natur. Math. **25** (1981).
8. R. Lidl and H. Niederreiter, *Finite fields*, Addison-Wesley, Reading, MA, 1983.
9. W. Ljunggren, *On the diophantine equation* $x^2 + D = y^n$, Norske Vid. Selsk. Forh. Trondheim **17** (1944), 37–43.
10. ——, *On a diophantine equation*, Norske Vid. Selsk. Forh. Trondheim **18** (1945), 125–128.
11. ——, *New theorems concerning the diophantine equation* $Cx^2 + D = y^n$, Norske Vid. Selsk. Forh. Trondheim **29** (1956), 1–4.
12. ——, *On the diophantine equation* $Cx^2 + D = y^n$, Pacific J. Math. **14** (1964), 585–596.
13. ——, *On the diophantine equation* $x^2 + D = 4y^9$, Monatsh. Math. **75** (1971), 136–143.
14. ——, *New theorems concerning the diophantine equation* $x^2 + D = 4y^n$, Acta Arith. **21** (1972), 183–191.
15. M. Mignotte and M. Waldschmidt, *Linear forms in two logarithms and Schneider's method*. III, Ann. Fac. Sci. Toulouse (1990), 43–75.
16. T. Nagell, *Sur l'impossibilité de quelques équation á deux indéterminées*, Norsk Mat. Forenings Skrifter Ser. 1 **13** (1923), 65–82.
17. ——, *Contributions to the theory of a category of diophantine equations of the second degree with two unknowns*, Nova Acta Soc. Sci. Upsal. (4) **16** (1955).
18. B. Persson, *On a diophantine equation in two unknowns*, Ark. Mat. **1** (1949), 45–57.
19. B. Stolt, *Die Anzahl von Lösungen gewisser diophantischer Gleichungen*, Arch. Math. **8** (1957), 393–400.

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