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# ON ABSOLUTE SUMMABILITY FACTORS

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ABSTRACT. In this paper a theorem on $|\overline{N}, p_n|_k$ summability factors, which generalizes a theorem of Mazhar (Indian J. Math. **14** (1972), 45–48) on $|C, 1|_k$ summability factors, has been proved.

## 1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with partial sums $(s_n)$. We denote the $n$th $(C, 1)$ means of the sequences $(s_n)$ and $(na_n)$ by $u_n$ and $t_n$, respectively. The series $\sum a_n$ is said to be summable $|C, 1|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n - u_{n-1}|^k < \infty. \tag{1.1}$$

But since $t_n = n(u_n - u_{n-1})$ (see [4]), condition (1.1) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{1.2}$$

Let $(p_n)$ be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as } n \to \infty \ (P_{-i} = p_{-i} = 0, \ i \geq 1). \tag{1.3}$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.4}$$

defines the sequence $(w_n)$ of the $(\overline{N}, p_n)$ mean of the sequence $(s_n)$ generated by the sequence of coefficients $(p_n)$ (see [3]). The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |w_n - w_{n-1}|^k < \infty.$$

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In the special case when $p_n = 1$ for all values of $n$ (resp. $k = 1$), $|\overline{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|\overline{N}, p_n|$) summability.

## 2. Mazhar's Theorem

Mazhar [5] has established the following theorem for $|C, 1|_k$ summability factors of infinite series.

**Theorem A.** *If $(X_n)$ is a positive monotonic nondecreasing sequence such that*

$$\lambda_m X_m = O(1) \quad \text{as } m \to \infty, \tag{2.1}$$

$$\sum_{n=1}^{m} n X_n |\Delta^2 \lambda_n| = O(1), \tag{2.2}$$

$$\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m) \quad \text{as } m \to \infty, \tag{2.3}$$

*then the series $\sum a_n \lambda_n$ is summable $|C, 1|_k$, $k \geq 1$.*

## 3. Main theorem

The aim of this paper is to generalize Theorem A for $|\overline{N}, p_n|_k$ summability in the form of the following theorem.

**Theorem.** *Let $(p_n)$ be a sequence of positive numbers such that*

$$P_n = O(n p_n) \quad \text{as } n \to \infty. \tag{3.1}$$

*If $(X_n)$ is a positive monotonic nondecreasing sequence such that conditions (2.1)–(2.2) of Theorem A are satisfied and*

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad \text{as } m \to \infty, \tag{3.2}$$

*then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$.*

It should be noted that if we take $p_n = 1$ for all values of $n$, then we get Theorem A. Also it may be noticed that under the conditions of the theorem, we have that

$$\Delta \lambda_n \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

## 4. Lemma

We need the following lemma for the proof of our theorem.

**Lemma.** *Under the conditions of the theorem we have that*

$$\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty, \tag{4.1}$$

$$n X_n |\Delta \lambda_n| = O(1) \quad \text{as } n \to \infty. \tag{4.2}$$

*Proof.* By (3.3), we can write

$$\Delta \lambda_n = \sum_{v=n}^{\infty} \Delta^2 \lambda_v .$$

Hence,

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| = \sum_{n=1}^{\infty} X_n\left|\sum_{v=n}^{\infty}\Delta^2\lambda v\right| \le \sum_{n=1}^{\infty} X_n \sum_{v=n}^{\infty}|\Delta^2\lambda_v|$$
$$= \sum_{v=1}^{\infty}|\Delta^2\lambda_v|\sum_{n=1}^{v} X_n \le \sum_{v=1}^{\infty} vX_v|\Delta^2\lambda_v| < \infty \quad \text{(by (2.2))}.$$

Since $(nX_n)$ is increasing, we have

$$nX_n|\Delta\lambda_n| = nX_n\left|\sum_{v=n}^{\infty}\Delta^2\lambda_v\right| \le nX_n\sum_{v=n}^{\infty}|\Delta^2\lambda_v|$$
$$\le \sum_{v=n}^{\infty} vX_v|\Delta^2\lambda_v| < \infty \quad \text{(by (2.2))}.$$

This complete the proof of the lemma.

## 5. Proof of the theorem

Let $(T_n)$ be the sequence of $(\overline{N}, p_n)$ means of the series $\sum a_n\lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r\lambda_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_v . \tag{5.1}$$

Then, for $n \ge 1$, we get

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v\lambda_v = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}\lambda_v}{v} va_v . \tag{5.2}$$

Applying Abel's transformation to the right-hand side of (5.2), we have

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\Delta\left(\frac{P_{v-1}\lambda_v}{v}\right)\sum_{r=1}^{v} ra_r + \frac{p_n\lambda_n}{nP_n}\sum_{v=1}^{n} va_v$$
$$= \frac{(n+1)p_nt_n\lambda_n}{nP_n} - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} p_vt_v\lambda_v\frac{v+1}{v}$$
$$+ \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v\Delta\lambda_vt_v\frac{v+1}{v} + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v\lambda_{v+1}t_v\frac{1}{v}$$
$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say}.$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4.$$

First, we have

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}|^k = O(1)\sum_{n=1}^{m}|\lambda_n|^{k-1}|\lambda_n|\frac{p_n}{P_n}|t_n|^k = O(1)\sum_{n=1}^{m}|\lambda_n|\frac{p_n}{P_n}|t_n|^k$$

$$= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\frac{p_v}{P_v}|t_v|^k + O(1)|\lambda_m|\sum_{v=1}^{m}\frac{p_v}{P_v}|t_v|^k$$

$$= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n + O(1)|\lambda_m|X_m = O(1) \quad \text{as } m\to\infty,$$

by virtue of the hypotheses and Lemma.

Now, applying Hölder's inequality, as in $T_{n,1}$, we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,2}|^k = O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{v=1}^{n-1}p_v|t_v|^k|\lambda_v|^k\right\}\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v|p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}|\lambda_v|\frac{p_v}{P_v}|t_v|^k = O(1) \quad \text{as } m\to\infty.$$

Again, using the fact that $P_v = O(vp_v)$, by (3.1), we get

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,3}|^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}P_v|\Delta\lambda_v|\,|t_v|\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}vp_v|\Delta\lambda_v|\,|t_v|\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{v=1}^{n-1}(v|\Delta\lambda_v|)^k p_v|t_v|^k\right\}\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m}(v|\Delta\lambda_v|)^{k-1}v|\Delta\lambda_v|p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{p_v}{P_v}|t_v|^k$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}\frac{p_r}{P_r}|t_r|^k + O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\frac{p_v}{P_v}|t_v|^k$$

$$= O(1)\sum_{v=1}^{m-1} |\Delta(v|\Delta\lambda_v|)|X_v + O(1)m|\Delta\lambda_m|X_m$$

$$= O(1)\sum_{v=1}^{m-1} vX_v|\Delta^2\lambda_v| + O(1)\sum_{v=1}^{m-1} |\Delta\lambda_{v+1}|X_v + O(1)m|\Delta\lambda_m|X_m$$

$$= O(1) \quad \text{as } m \to \infty,$$

by virtue of the hypotheses and Lemma.

Finally, using the fact that $P_v = O(vp_v)$, by (3.1), as in $T_{n,1}$, we have that

$$\sum_{n=1}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,4}|^k = O(1)\sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{\sum_{v=1}^{n-1} |\lambda_{v+1}| p_v |t_v|\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{\sum_{v=1}^{n-1} |\lambda_{v+1}|^k p_v |t_v|^k\right\} \times \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m} |\lambda_{v+1}|^{k-1} |\lambda_{v+1}| p_v |t_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m} |\lambda_{v+1}| \frac{p_v}{P_v} |t_v|^k = O(1) \quad \text{as } m \to \infty.$$

Therefore, we get that

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,r}|^k = O(1) \quad \text{as } m \to \infty, \quad \text{for } r = 1, 2, 3, 4.$$

This completes the proof of the theorem.

## References


1. H. Bor, *On two summability methods*, Math. Proc. Cambridge Philos. Soc. **97** (1985), 147–149.
2. T. M. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. London Math. Soc. **7** (1957), 113–141.
3. G. H. Hardy, *Divergent series*, Oxford Univ. Press, New York and London, 1949.
4. E. Kogbetliantz, *Sur les séries absolument sommables par la méthode des moyennes arithmétiques*, Bull. Sci. Mat. **49** (1925), 234–256.
5. S. M. Mazhar, *On* $|C, 1|_k$ *summability factors of infinite series*, Indian J. Math. **14** (1972), 45–48.



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