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# THE SINGLE-VALUED EXTENSION PROPERTY AND SPECTRAL MANIFOLDS

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ABSTRACT. We discuss the relation between the single-valued extension property (that is, Dunford's property (A)) and spectral manifolds $X_T(F)$ of a bounded linear operator. In particular, we prove that Dunford's property (C) implies the property (A). We also prove that if $T \in B(X)$ has the property $(\beta^*)$ introduced by Fong, then $X^*_{T^*}(F) = X_T(\mathbb{C}\backslash F)^{\perp}$ for every closed set $F$ in the complex plane $\mathbb{C}$.

In the spectral decomposition theory of bounded linear operators the single-valued extension property is an elementary and important property. All spectral, decomposable, and hyponormal operators have this property, but there are some ordinary operators which do not have this property, for example, the adjoint of the unilateral shift on a Hilbert space. For an operator $T \in B(X)$ without the single-valued extension property we may also define the spectral manifold $X_T(F)$. In this case the properties of $X_T(F)$ are different than ones of operators with the single-valued extension property. In the first part of this paper we discuss some dependent relations between the single-valued extension property and spectral manifolds $X_T(F)$. In particular, we prove that Dunford's property (C) implies his property (A), that is, the single-valued extension property, and so we strengthen an essential and important result in the theory of spectral operators due to Dunford and Schwartz [2]. In the second part of the paper we prove that if for every open covering $\{G_1, \dots, G_n\}$ of $\sigma(T)$,

$$X = X_T(\overline{G}_1) + \cdots + X_T(\overline{G}_n),$$

then $X^*_{T^*}(F) = X_T(\mathbb{C}\backslash F)^{\perp}$ for each closed set $F$ in the complex plane $\mathbb{C}$. So we generalize and deepen a main result in the duality theory due to Frunză [4].

In this paper $\mathbb{C}$ denotes the complex plane, $X$ the complex Banach space, and $B(X)$ the Banach algebra of all bounded linear operators on $X$. If $T \in B(X)$ and $F$ is a closed set in $\mathbb{C}$, we define

$$X_T(F) = \{x \in X\,;\ \text{there exists an analytic } X\text{-valued function } f\colon \mathbb{C}\backslash F \to X \text{ such that } (\lambda - T)f(\lambda) = x,\ \lambda \in \mathbb{C}\backslash F\}.$$

$X_T(F)$ is said to be the spectral manifold of $T$. If $G$ is an open set in $\mathbb{C}$, we

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define

$$X_T(G) = \bigcup\{X_T(F);\ F \subseteq G \text{ and } F \text{ is closed}\},$$

where "$\bigcup$" denotes the linear span. If $T$ has the single-valued extension property, then the above definition is identical with Dunford and Schwartz's original definition

$$X_T(F) = \{x \in X;\ \sigma_T(x) \subseteq F\},$$

where $\sigma_T(x)$ is the local spectrum of $T$ at $x$.

## 1

It is well known that if $T \in B(X)$ has the single-valued extension property, then the following propositions are true:

(1) For arbitrary closed sets $F_1$ and $F_2$ in $\mathbb{C}$,

$$X_T(F_1) \cap X_T(F_2) = X_T(F_1 \cap F_2).$$

(2) If $X_T(F)$ is closed for a closed set $F$ in $\mathbb{C}$, then

$$\sigma(T|X_T(F)) \subseteq \sigma(T) \cap F.$$

(3) If $X_T(F)$ is closed for a closed set $F$ in $\mathbb{C}$, then $X_T(F)$ is an analytic invariant subspace of $T$, that is, $X_T(F)$ is an invariant subspace of $T$ and $f(\lambda) \in X_T(F)$ for an arbitrary analytic $X$-valued function $f\colon G \to X$ on some open set $G$ in $\mathbb{C}$ satisfying $(\lambda - T)f(\lambda) \in X_T(F)$, $\lambda \in G$.

(4) If $X_T(F)$ is closed for a closed set $F$ in $\mathbb{C}$, then $X_T(F)$ is the spectral maximal space of $T$, that is, $X_T(F)$ is an invariant subspace of $T$ and for an arbitrary invariant subspace $Y$ of $T$,

$$\sigma(T|Y) \subseteq \sigma(T|X_T(F)) \quad \text{implies} \quad Y \subseteq X_T(F).$$

If $T$ does not have the single-valued extension property, then these propositions are false.

**Example 1.** Let $H$ be a separable complex Hilbert space, $\{e_n\}_{n=-\infty}^{+\infty}$ the orthonormal basis of $H$, and $T \in B(H)$ a bilateral weighted shift on $H$:

$$Te_n = w_n e_{n+1}, \qquad n = 0, \pm 1, \pm 2, \ldots,$$

where $\{w_n\}_{n=-\infty}^{+\infty}$ is a monotone decreasing sequence of positive numbers and $w_n \nearrow 1$ as $n \to -\infty$, $w_n \searrow \frac{1}{2}$ as $n \to +\infty$. It follows from [5] that $T^*$ is a hyponormal operator and

$$\sigma(T) = \{\lambda \in \mathbb{C};\ \tfrac{1}{2} \le |\lambda| \le 1\}.$$

It follows from [6] that $T$ does not have the single-valued extension property. We write

$$F_1 = \{\lambda \in \mathbb{C};\ |\lambda| = 1\}, \qquad F_2 = \{\lambda \in \mathbb{C};\ |\lambda| = \tfrac{1}{2}\},$$

$$H_k^+ = \bigvee_{n=k}^{+\infty}\{e_n\}, \qquad H_k^- = \bigvee_{n=-\infty}^{k}\{e_n\},$$

where "$\bigvee$" denotes the closed linear span.

**Proposition 1.** *For arbitrary integer $k$,*

$$H_k^- \subseteq X_T(F_1), \qquad H_k^+ \subseteq X_T(F_2).$$

*Proof*. Since $H_k^+$ is an invariant subspace of $T$, $T|H_k^+$ is a unilateral weighted shift with the weight sequence $\{w_n\}_{n=k}^{+\infty}$, and $w_n \searrow \frac{1}{2}$ as $n \to +\infty$, it follows from [5] that

$$\sigma(T|H_k^+) = \{\lambda \in \mathbb{C};\ |\lambda| \leq \tfrac{1}{2}\}.$$

For arbitrary $x \in H_k^+$, $(\lambda - T|H_k^+)^{-1}x$ is an analytic $X$-valued function on $\{\lambda \in \mathbb{C};\ |\lambda| > \frac{1}{2}\}$ and

$$(\lambda - T)(\lambda - T|H_k^+)^{-1}x = x, \qquad |\lambda| > \tfrac{1}{2}.$$

Because $\sigma(T) = \{\lambda \in \mathbb{C};\ \frac{1}{2} \leq |\lambda| \leq 1\}$, we can define

$$f_x(\lambda) = \begin{cases} (\lambda - T)^{-1}x, & |\lambda| < \frac{1}{2}, \\ (\lambda - T|H_k^+)^{-1}x, & |\lambda| > \frac{1}{2}; \end{cases}$$

then $f_x(\lambda)$ is an analytic $X$-valued function on $\mathbb{C}\backslash F_2$ and

$$(\lambda - T)f_x(\lambda) = x, \qquad \lambda \in \mathbb{C}\backslash F_2.$$

So $x \in X_T(F_2)$, that is, $H_k^+ \subseteq X_T(F_2)$.

For the first inclusion we note that $T$ is invertible and

$$T^{-1}e_n = w_{n-1}^{-1}e_{n-1}, \qquad n = 0, \pm 1, \pm 2, \dots,$$

so for arbitrary integer $k$, $H_k^-$ is an invariant subspace of $T^{-1}$. Analogously we can prove $H_k^- \subseteq X_{T^{-1}}(F_1)$. It follows from the definition of the spectral manifold that $X_{T^{-1}}(F_1) = X_T(F_1)$, since $F_1$ is the unit circle. Hence $H_k^- \subseteq X_T(F_1)$, so the proof is complete.

It follows from Proposition 1 that $X_T(F_1) \cap X_T(F_2) \neq \{0\}$. But $X_T(F_1 \cap F_2) = \{0\}$, since $F_1 \cap F_2 = \varnothing$. Therefore statement (1) is false.

Choosing $F = F_1 \cup F_2$, it follows from Proposition 1 that

$$X_T(F) = X_T(F_1) + X_T(F_2) = H.$$

Hence $\sigma(T|X_T(F)) = \sigma(T) \nsubseteq F$, that is, statement (2) is false.

For statement (3) we choose $F = \varnothing$. Then $X_T(F) = \{0\}$ is not an analytic invariant subspace of $T$.

**Example 2.** Let $T_1$ be the bilateral weighted shift in Example 1, $T_2 = \frac{2}{3}U$, where $U$ is the bilateral shift (nonweighted) on $H$, and $X = H \oplus H$, $T = T_1 \oplus T_2$, $F = F_1 \cup F_2$, where $F_1$, $F_2$ are the closed sets in Example 1. Then $T \in B(X)$ does not have the single-valued extension property, $X_T(F) = H \oplus \{0\}$, and $\sigma(T|X_T(F)) = \sigma(T_1)$. Note that $\sigma(T) = \sigma(T_1) \cup \sigma(T_2) \subseteq \sigma(T_1)$, that is, $\sigma(T) \subseteq \sigma(T|X_T(F))$, but $X = H \oplus H \supsetneq X_T(F)$. Hence $X_T(F)$ is not the spectral maximal space of $T$.

Regardless of whether a bounded linear operator $T$ has the single-valued extension property, the following propositions are true.

**Proposition 2.** *For an arbitrary closed set $F$ in $\mathbb{C}$, $X_T(F)$ is a hyperinvariant manifold of $T$. In particular, if $X_T(F)$ is closed, then it is a hyperinvariant subspace of $T$.*

*Proof*. Suppose $x \in X_T(F)$. Then there exists an analytic $X$-valued function $f\colon \mathbb{C}\backslash F \to X$ such that

$$(\lambda - T)f(\lambda) = x, \qquad \lambda \in \mathbb{C}\backslash F.$$

So $Sf(\lambda)$ is also an analytic $X$-valued function on $\mathbb{C}\backslash F$ for each $S \in B(X)$ commuting with $T$, and

$$(\lambda - T)Sf(\lambda) = S(\lambda - T)f(\lambda) = Sx, \qquad \lambda \in \mathbb{C}\backslash F.$$

Hence $Sx \in X_T(F)$, so the proof is complete.

**Proposition 3.** *If $X_T(F)$ is closed for a closed set $F$ in $\mathbb{C}$, then*

$$\sigma(T|X_T(F)) \subseteq \sigma(T) \cap \widehat{F},$$

*where $\widehat{F}$ is the union of $F$ and all bounded components in $\mathbb{C}\backslash F$.*

*Proof.* Suppose $\lambda_0 \notin \widehat{F}$. To show $\lambda_0 \notin \sigma(T|X_T(F))$, it is sufficient to prove that $\lambda_0 - T|X_T(F)$ is bijective. According to the definition of $X_T(F)$, it is surjective. If $x \in X_T(F)$ satisfies $(\lambda_0 - T)x = 0$, then there exists an analytic $X$-valued function $f\colon \mathbb{C}\backslash F \to X$ such that

$$(\lambda - T)f(\lambda) = x, \qquad \lambda \in \mathbb{C}\backslash F.$$

Let

$$g(\lambda) = \frac{1}{\lambda - \lambda_0}x, \qquad \lambda \neq \lambda_0.$$

Then $g(\lambda)$ is an analytic $X$-valued function on $\{\lambda \in \mathbb{C};\ \lambda \neq \lambda_0\}$ and

$$(\lambda - T)g(\lambda) = x, \qquad \lambda \neq \lambda_0.$$

We define

$$G(\lambda) = \begin{cases} f(\lambda), & \lambda \in \mathbb{C}\backslash\widehat{F}, \\ g(\lambda), & \lambda \neq \lambda_0. \end{cases}$$

Note that each component of $\mathbb{C}\backslash\widehat{F}$ is connected with $\rho(T)$, the resolvent set of $T$, and both $f(\lambda)$ and $g(\lambda)$ are analytic continuations of $R(\lambda, T)x$, $\lambda \in \rho(T)$; hence

$$f(\lambda) = g(\lambda), \qquad \lambda \in (\mathbb{C}\backslash\widehat{F})\backslash\{\lambda_0\}.$$

It shows that $G(\lambda)$ is well defined. Because $G(\lambda)$ is analytic on the whole complex plane and

$$(\lambda - T)G(\lambda) = x, \qquad \lambda \in \mathbb{C},$$

$x \in X_T(\varnothing) = \{0\}$, that is, $x = 0$. The proof is complete.

**Theorem 1.** *Suppose that $F_1$ and $F_2$ are two disjoint closed sets in $\mathbb{C}$. Then*

$$X_T(F_1 \cup F_2) = X_T(F_1) + X_T(F_2).$$

*Proof.* We may assume without loss of generality that $F_1 \cup F_2 \subseteq \sigma(T)$. If $x \in X_T(F_1 \cup F_2)$, then there exists an analytic $X$-valued function $f\colon \mathbb{C}\backslash(F_1 \cup F_2) \to X$ such that

$$(\lambda - T)f(\lambda) = x, \qquad \lambda \in \mathbb{C}\backslash(F_1 \cup F_2).$$

For two arbitrary disjoint smoothed closed curves $C_1$ and $C_2$ in the interior of the curve $C = \{\lambda \in \mathbb{C};\ |\lambda| = \|T\| + 1\}$ such that $C_j$ surrounds $F_j$, $j = 1, 2$, respectively, we have

$$\begin{aligned} x &= \frac{1}{2\pi i}\int_C (\lambda - T)^{-1}x\, d\lambda = \frac{1}{2\pi i}\int_C f(\lambda)\, d\lambda = \frac{1}{2\pi i}\int_{C_1 \cup C_2} f(\lambda)\, d\lambda \\ &= \frac{1}{2\pi i}\int_{C_1} f(\lambda)\, d\lambda + \frac{1}{2\pi i}\int_{C_2} f(\lambda)\, d\lambda. \end{aligned}$$

Let $x_j = (1/2\pi i)\int_{C_j} f(\lambda)\,d\lambda$. Because $f(\lambda)$ is analytic, the $x_j$ are independent of the selection of the curves $C_j$, $j = 1, 2$. It is sufficient to prove $x_j \in X_T(F_j)$, $j = 1, 2$.

For arbitrary $\mu \in \mathbb{C}\backslash F_1$ there exists a smoothed closed curve $J$ surrounding $F_1$ such that both $\mu$ and $F_2$ are outside $J$. Let

$$g(\mu) = -\frac{1}{2\pi i}\int_J \frac{f(\lambda)}{\lambda - \mu}\,d\lambda.$$

Then $g(\mu)$ is independent of the selection of the curve $J$. It is easily seen that the function $g$ is analytic on $\mathbb{C}\backslash F_1$ and

$$\begin{aligned}(\mu - T)g(\mu) &= \frac{1}{2\pi i}\int_J \frac{(T-\mu)f(\lambda)}{\lambda - \mu}\,d\lambda \\ &= \frac{1}{2\pi i}\int_J \frac{(T-\lambda)f(\lambda)}{\lambda - \mu}\,d\lambda + \frac{1}{2\pi i}\int_J f(\lambda)\,d\lambda \\ &= \frac{1}{2\pi i}\int_J \frac{-x}{\lambda-\mu}\,d\lambda + x_1 = x_1, \qquad \mu \in \mathbb{C}\backslash F_1.\end{aligned}$$

So $x_1 \in X_T(F_1)$. The proof that $x_2 \in X_T(F_2)$ is analogous. The proof is complete.

In the case that $T$ has the single-valued extension property and $X_T(F_1 \cup F_2)$ is closed, this result is well known. Now we have proved the result without any assumption.

**Theorem 2.** *Suppose that $T \in B(X)$ and $X_T(F)$ is closed for every closed set $F$ in $\mathbb{C}$. Then $T$ has the single-valued extension property.*

*Proof.* Let $f\colon G \to X$ be an analytic $X$-valued function on some open set $G$ in $\mathbb{C}$ such that

$$(\lambda - T)f(\lambda) = 0, \qquad \lambda \in G.$$

We may assume without loss of generality that $G$ is connected. Choose an open disc $U$ in $G$ such that $\overline{U} \subseteq G$; then $\widehat{\overline{U}} = \overline{U}$. According to the supposition of the theorem, $X_T(\overline{U})$ is closed, it follows from Proposition 3 that

$$\sigma(T|X_T(\overline{U})) \subseteq \overline{U}.$$

For arbitrary $\lambda_0 \in U$ let $x_0(\lambda) = f(\lambda_0)/(\lambda - \lambda_0)$, $\lambda \neq \lambda_0$. Then $x_0(\lambda)$ is analytic on $\mathbb{C}\backslash\{\lambda_0\}$ and

$$(\lambda - T)x_0(\lambda) = f(\lambda_0), \qquad \lambda \neq \lambda_0.$$

Hence $f(\lambda_0) \in X_T(\{\lambda_0\}) \subseteq X_T(\overline{U})$. Because $G$ is connected and $X_T(\overline{U})$ is closed, it follows by means of analytic continuation that

$$f(\lambda) \in X_T(\overline{U}), \qquad \lambda \in G.$$

Hence we have

$$(\lambda - T|X_T(\overline{U}))f(\lambda) = 0, \qquad \lambda \in G.$$

If $\lambda \in G\backslash\overline{U}$, then $\lambda \notin \sigma(T|X_T(\overline{U}))$; thus $f(\lambda) = 0$. It follows again by means of analytic continuation that

$$f(\lambda) = 0, \qquad \lambda \in G.$$

The proof is complete.

From Theorem 2 we know that Dunford's property (C) implies his property (A). In the theory of spectral operators due to Dunford and Schwartz [2] there is the following essential and important result.

**Theorem D-S.** *Let $T$ be a bounded linear operator in a weakly complete space. Then $T$ is a spectral operator if and only if $T$ has properties* (A), (B), (C), *and* (D).

According to our Theorem 2 the property (A) in this theorem may be dropped, and this theorem can be strengthened as follows.

**Theorem 3.** *Let $T$ be a bounded linear operator in a weakly complete space. Then $T$ is a spectral operator if and only if $T$ has properties* (B), (C), *and* (D).

## 2

Frunză [4] proved that if $T \in B(X)$ is decomposable, then for every closed set $F$ in $\mathbb{C}$

$$X^*_{T^*}(F) = X_T(\mathbb{C}\backslash F)^{\perp},$$

where

$$X_T(\mathbb{C}\backslash F)^{\perp} = \{u \in X^*;\ \langle x, u\rangle = 0, \text{ for each } x \in X_T(\mathbb{C}\backslash F)\}.$$

In the following we will prove this result under a weaker condition. First we introduce the following notion due to Fong: $T \in B(X)$ is said to have the property $(\beta^*)$ if for every open covering $\{G_1, \dots, G_n\}$ of $\sigma(T)$,

$$X = X_T(\overline{G}_1) + \cdots + X_T(\overline{G}_n).$$

Here $X_T(\overline{G}_k)$, $k = 1, \dots, n$, are not necessarily closed. If, in addition, all $X_T(\overline{G}_k)$, $k = 1, \dots, n$, are closed, then $T$ is decomposable.

Fong [3] proved that the property $(\beta^*)$ is the duality property of Bishop's property $(\beta)$, that is, $T \in B(X)$ has the property $(\beta)$ if and only if $T^*$, the adjoint of $T$, has the property $(\beta^*)$. It is easily known that all decomposable operators have properties $(\beta)$ and $(\beta^*)$. The operator $T$ in Example 1 has the property $(\beta^*)$, but it is not decomposable. In fact, because $T^*$ is hyponormal, $T^*$ has the property $(\beta)$. It follows from the result due to Fong that $T$ has the property $(\beta^*)$. In addition, $T$ is not decomposable, since $T$ does not have the single-valued extension property. Thus the property $(\beta^*)$ is weaker than decomposable. The following theorem is a nontrivial generalization of the above result due to Frunză.

**Theorem 4.** *Let $T \in B(X)$ have the property $(\beta^*)$. Then for every closed set $F$ in $\mathbb{C}$*

$$X^*_{T^*}(F) = X_T(\mathbb{C}\backslash F)^{\perp}.$$

*Proof*. First we prove $X_T(\mathbb{C}\backslash F)^{\perp} \subseteq X^*_{T^*}(F)$. Because $X_T(\mathbb{C}\backslash F)^{\perp}$ is an invariant subspace of $T^*$, it is sufficient to prove

$$\sigma(T^*|X_T(\mathbb{C}\backslash F)^{\perp}) \subseteq F.$$

Let $\lambda_0 \in \mathbb{C}\backslash F$. It is easy to show that $\lambda_0 - T^*|X_T(\mathbb{C}\backslash F)^{\perp}$ is injective. In fact, if $u \in X_T(\mathbb{C}\backslash F)^{\perp}$ such that $(\lambda_0 - T^*)u = 0$, then we can show $u = 0$ as follows: Suppose that $\{G_1, G_2\}$ is an open covering of $\sigma(T)$ such that

$$\lambda_0 \notin \overline{G}_1, \qquad \lambda_0 \in G_2 \subseteq \overline{G}_2 \subseteq \mathbb{C}\backslash F.$$

According to the property $(\beta^*)$ of $T$,

$$X = X_T(\overline{G}_1) + X_T(\overline{G}_2).$$

Thus for each $x \in X$ there are $x_k \in X_T(\overline{G}_k)$, $k = 1, 2$, such that

$$x = x_1 + x_2.$$

Because $x_1 \in X_T(\overline{G}_1)$, there exists an analytic $X$-valued function $f_{x_1}(\lambda)$ on $\mathbb{C}\backslash\overline{G}_1$ such that

$$x_1 = (\lambda - T)f_{x_1}(\lambda), \qquad \lambda \in \mathbb{C}\backslash\overline{G}_1.$$

Note that $x_2 \in X_T(\overline{G}_2) \subseteq X_T(\mathbb{C}\backslash F)$ and $\lambda_0 \in \mathbb{C}\backslash\overline{G}_1$. Then we have

$$\begin{aligned}\langle x, u\rangle &= \langle x_1 + x_2, u\rangle = \langle x_1, u\rangle = \langle(\lambda_0 - T)f_{x_1}(\lambda_0), u\rangle \\ &= \langle f_{x_1}(\lambda_0), (\lambda_0 - T^*)u\rangle = 0.\end{aligned}$$

Hence $u = 0$.

Before proving that $\lambda_0 - T^*|X_T(\mathbb{C}\backslash F)^\perp$ is surjective we give some primary knowledge. Choose an open covering $\{G_1, G_2\}$ of $\mathbb{C}$ such that

$$\lambda_0 \notin \overline{G}_1, \qquad \lambda_0 \in G_2 \subseteq \overline{G}_2 \subseteq \mathbb{C}\backslash F.$$

Choose again another open covering $\{D_1, D_2\}$ of $\mathbb{C}$ which satisfies

$$F \subseteq D_1 \subseteq \overline{D}_1 \subseteq G_1, \qquad \lambda_0 \in D_2 \subseteq \overline{D}_2 \subseteq G_2.$$

Because $T$ has the property $(\beta^*)$, we have

$$X = X_T(\overline{D}_1) + X_T(\overline{D}_2) = X_T(\overline{G}_1) + X_T(\overline{G}_2). \tag{1}$$

Since the above four manifolds $X_T(\cdot)$ are not necessarily closed, we need the following discussion. For an arbitrary closed set $E$ in $\mathbb{C}$, let

$$X_T^b(E) = \{x \in X;\ \text{there exists a uniformly bounded analytic } X\text{-valued function } f_x \text{ on } \mathbb{C}\backslash E \text{ such that } (\lambda - T)f_x(\lambda) = x,\ \lambda \in \mathbb{C}\backslash E\}.$$

For $x \in X_T^b(E)$ we define

$$\|x\|_b^E = \inf\{\sup\{\|f_x(\lambda)\|;\ \lambda \in \mathbb{C}\backslash E\};\ f_x \text{ is a uniformly bounded analytic } X\text{-valued function on } \mathbb{C}\backslash E \text{ such that } (\lambda - T)f_x(\lambda) = x,\ \lambda \in \mathbb{C}\backslash E\}.$$

It is easy to verify that $\|\cdot\|_b^E$ is a norm on $X_T^b(E)$ and $X_T^b(E)$ is complete under this norm; hence it is a Banach space. For a fixed $\lambda_1 \in \mathbb{C}\backslash E$, for every $x \in X_T^b(E)$,

$$\begin{aligned}\|x\| &= \|(\lambda_1 - T)f_x(\lambda_1)\| \le \|\lambda_1 - T\|\,\|f_x(\lambda_1)\| \\ &\le \|\lambda_1 - T\| \sup\{\|f_x(\lambda)\|;\ \lambda \in \mathbb{C}\backslash E\}.\end{aligned}$$

So $\|x\| \le \|\lambda_1 - T\|\,\|x\|_b^E$, that is, there is a constant $M_E$ such that $\|x\| \le M_E\|x\|_b^E$ for every $x \in X_T^b(E)$. According to the selection of $\{G_1, G_2\}$ and $\{D_1, D_1\}$, we have

$$X_T(\overline{D}_k) \subseteq X_T^b(\overline{G}_k) \subseteq X_T(\overline{G}_k), \qquad k = 1, 2.$$

It follows from (1) that

$$X = X_T^b(\overline{G}_1) + X_T^b(\overline{G}_2). \tag{2}$$

We introduce a new Banach space

$$X^b = X_T^b(\overline{G}_1) \oplus X_T^b(\overline{G}_2).$$

The norm in $X^b$ is defined as

$$\|x\|_b = \|x_1\|_b^1 + \|x_2\|_b^2$$

for $x = x_1 \oplus x_2$, $x_k \in X_T^b(\overline{G}_k)$, $k = 1, 2$, where $\|x_j\|_b^j = \|x_j\|_b^{\overline{G}_j}$, $j = 1, 2$.

Let $J: X^b \to X$ be defined as

$$Jx = x_1 + x_2$$

for $x = x_1 \oplus x_2$, $x_k \in X_T^b(\overline{G}_k)$, $k = 1, 2$. Evidently, $J$ is linear and

$$\begin{aligned}\|Jx\| &= \|x_1 + x_2\| \le \|x_1\| + \|x_2\| \\ &\le M_{\overline{G}_1}\|x_1\|_b^1 + M_{\overline{G}_2}\|x_2\|_b^2 \le M\|x\|_b,\end{aligned}$$

where $M = \max\{M_{\overline{G}_1}, M_{\overline{G}_2}\}$. This shows that $J$ is a bounded linear map from $X^b$ to $X$. It follows from (2) that $J$ is surjective. According to the open map theorem there is a constant $M_b > 0$ such that for every $x \in X$ there are $x_k \in X_T^b(\overline{G}_k)$, $k = 1, 2$, satisfying

$$x = x_1 \oplus x_2 \quad \text{and} \quad \|x_k\|_b^k \le M_b\|x\|, \qquad k = 1, 2.$$

Now we can prove that $\lambda_0 - T^*|X_T(\mathbb{C}\backslash F)^\perp$ is surjective. Suppose $u \in X_T(\mathbb{C}\backslash F)^\perp$. We define $v$ as follows: For every $x \in X$ there are $x_k \in X_T^b(\overline{G}_k)$, $k = 1, 2$, such that $x = x_1 + x_2$. Because $x_1 \in X_T^b(\overline{G}_1)$, there exists a uniformly bounded analytic $X$-valued function $f_{x_1}$ on $\mathbb{C}\backslash\overline{G}_1$ such that

$$(\lambda - T)f_{x_1}(\lambda) = x_1, \qquad \lambda \in \mathbb{C}\backslash\overline{G}_1.$$

Note that $\lambda_0 \in \mathbb{C}\backslash\overline{G}_1$. We can define

$$\langle x, v\rangle = \langle f_{x_1}(\lambda_0), u\rangle;$$

$v$ is well defined. In fact, suppose that there again are $y_k \in X_T^b(\overline{G}_k)$, $k = 1, 2$, such that $x = y_1 + y_2$ and another uniformly bounded analytic $X$-valued function $f_{y_1}$ on $\mathbb{C}\backslash\overline{G}_1$ such that

$$(\lambda - T)f_{y_1}(\lambda) = y_1, \qquad \lambda \in \mathbb{C}\backslash\overline{G}_1.$$

(Note: it is possible that $x_k = y_k$, $k = 1, 2$, but $f_{x_1}$ and $f_{y_1}$ are different.) Then

$$y_1 - x_1 = x_2 - y_2 \in X_T^b(\overline{G}_1) \cap X_T^b(\overline{G}_2) \subseteq X_T(\overline{G}_1) \cap X_T(\overline{G}_2).$$

Note that $f_{y_1} - f_{x_1}$ is an analytic $X$-valued function on $\mathbb{C}\backslash\overline{G}_1$ and

$$(\lambda - T)(f_{y_1}(\lambda) - f_{x_1}(\lambda)) = y_1 - x_1, \qquad \lambda \in \mathbb{C}\backslash\overline{G}_1.$$

Because $y_1 - x_1 \in X_T(\overline{G}_2)$, there exists an analytic $X$-valued function $g$ on $\mathbb{C}\backslash\overline{G}_2$ such that

$$(\lambda - T)g(\lambda) = y_1 - x_1, \qquad \lambda \in \mathbb{C}\backslash\overline{G}_2.$$

Since $\{G_1, G_2\}$ is an open covering of $\mathbb{C}$, $(\mathbb{C}\backslash\overline{G}_1) \cap (\mathbb{C}\backslash\overline{G}_2) = \varnothing$. So we can define

$$G(\lambda) = \begin{cases} f_{y_1}(\lambda) - f_{x_1}(\lambda), & \lambda \in \mathbb{C}\backslash\overline{G}_1, \\ g(\lambda), & \lambda \in \mathbb{C}\backslash\overline{G}_2. \end{cases}$$

Then $G(\lambda)$ is an analytic $X$-valued function on $(\mathbb{C}\backslash\overline{G}_1)\cup(\mathbb{C}\backslash\overline{G}_2) = \mathbb{C}\backslash(\overline{G}_1\cap\overline{G}_2)$, and

$$(\lambda - T)G(\lambda) = y_1 - x_1, \qquad \lambda \in \mathbb{C}\backslash(\overline{G}_1 \cap \overline{G}_2).$$

This shows that $y_1 - x_1 \in X_T(\overline{G}_1 \cap \overline{G}_2)$, so $G(\lambda_0) = f_{y_1}(\lambda_0) - f_{x_1}(\lambda_0) \in X_T(\overline{G}_1 \cap \overline{G}_2) \subseteq X_T(\mathbb{C}\backslash F)$. Therefore

$$\langle f_{y_1}(\lambda_0) - f_{x_1}(\lambda_0), u\rangle = 0,$$

that is,

$$\langle f_{y_1}(\lambda_0), u\rangle = \langle f_{x_1}(\lambda_0), u\rangle.$$

This shows that $v$ is well defined.

Evidently, $v$ is linear. According to the above argument, there is a constant $M_b > 0$ such that for every $x \in X$ there are $x_k \in X_T^b(\overline{G}_k)$, $k = 1, 2$, satisfying $x = x_1 + x_2$, $\|x_k\|_b^k \le M_b\|x\|$, $k = 1, 2$. So

$$\begin{aligned} |\langle x, v\rangle| &= |\langle f_{x_1}(\lambda_0), u\rangle| \le \|f_{x_1}(\lambda_0)\| \, \|u\| \\ &\le \sup\{\|f_{x_1}(\lambda)\|;\ \lambda \in \mathbb{C}\backslash\overline{G}_1\}\|u\|. \end{aligned}$$

Thus

$$|\langle x, v\rangle| \le \|x_1\|_b\|u\| \le M_b\|x\| \, \|u\|.$$

This shows that $v$ is bounded, that is, $v \in X^*$.

Suppose $x \in X_T(\mathbb{C}\backslash F)$. It follows from (2) that there are $x_k \in X_T^b(\overline{G}_k) \subseteq X_T(\overline{G}_k)$, $k = 1, 2$, such that $x = x_1 + x_2$. Thus $x_1 = x - x_2 \in X_T(\mathbb{C}\backslash F)$. Then there is a closed set $E \subseteq \mathbb{C}\backslash F$ such that $x_1 \in X_T(E)$. Noting $\mathbb{C}\backslash G_1 \subseteq \mathbb{C}\backslash F$, we may assume without loss of generality that $E \supseteq \mathbb{C}\backslash G_1$. Let $g: \mathbb{C}\backslash E \to X$ be an analytic $X$-valued function such that

$$(\lambda - T)g(\lambda) = x_1, \qquad \lambda \in \mathbb{C}\backslash E.$$

There also exists an analytic $X$-valued function $f_{x_1}$ on $\mathbb{C}\backslash\overline{G}_1$ such that

$$(\lambda - T)f_{x_1}(\lambda) = x_1, \qquad \lambda \in \mathbb{C}\backslash\overline{G}_1.$$

Noting that $\mathbb{C}\backslash\overline{G}_1$ and $\mathbb{C}\backslash E$ are disjoint, we can define an analytic $X$-valued function on $(\mathbb{C}\backslash\overline{G}_1) \cup (\mathbb{C}\backslash E) = \mathbb{C}\backslash(\overline{G}_1 \cap E)$ as

$$G(\lambda) = \begin{cases} f_{x_1}(\lambda), & \lambda \in \mathbb{C}\backslash\overline{G}_1, \\ g(\lambda), & \lambda \in \mathbb{C}\backslash E, \end{cases}$$

and it satisfies

$$(\lambda - T)G(\lambda) = x_1, \qquad \lambda \in \mathbb{C}\backslash(\overline{G}_1 \cap E).$$

So $x_1 \in X_T(\overline{G}_1 \cap E)$. Then $G(\lambda_0) = f_{x_1}(\lambda_0) \in X_T(\overline{G}_1 \cap E) \subseteq X_T(\mathbb{C}\backslash F)$. Hence

$$\langle x, v\rangle = \langle f_{x_1}(\lambda_0), u\rangle = 0.$$

This shows $v \in X_T(\mathbb{C}\backslash F)^\perp$.

For every $x \in X$ there are $x_k \in X_T^b(\overline{G}_k)$, $k = 1, 2$, such that $x = x_1 + x_2$, and there exists an analytic $X$-valued function $f_{x_1}$ on $\mathbb{C}\backslash\overline{G}_1$ such that

$$(\lambda - T)f_{x_1}(\lambda) = x_1, \qquad \lambda \in \mathbb{C}\backslash\overline{G}_1.$$

Note that $(\lambda_0 - T)x_k \in X_T^b(\overline{G}_k)$, $k = 1, 2$, such that

$$(\lambda_0 - T)x = (\lambda_0 - T)x_1 + (\lambda_0 - T)x_2$$

and that $(\lambda_0 - T)f_{x_1}(\lambda)$ is also an analytic $X$-valued function on $\mathbb{C}\backslash\overline{G}_1$ such that

$$(\lambda - T)(\lambda_0 - T)f_{x_1}(\lambda) = (\lambda_0 - T)x_1, \qquad \lambda \in \mathbb{C}\backslash\overline{G}_1.$$

So

$$\begin{aligned}\langle x, (\lambda_0 - T^*)v\rangle &= \langle(\lambda_0 - T)x, v\rangle = \langle(\lambda_0 - T)f_{x_1}(\lambda_0), u\rangle \\ &= \langle x_1, u\rangle = \langle x_1 + x_2, u\rangle = \langle x, u\rangle.\end{aligned}$$

Because $x$ is arbitrary, $(\lambda_0 - T^*)v = u$, that is, $\lambda_0 - T^*|X_T(\mathbb{C}\backslash F)^\perp$ is surjective.

Finally, we prove $X_{T^*}^*(F) \subseteq X_T(\mathbb{C}\backslash F)^\perp$. Suppose $u \in X_{T^*}^*(F)$. Then there exists an analytic $X^*$-valued function $f_u^*: \mathbb{C}\backslash F \to X^*$ such that

$$(\lambda - T^*)f_u^*(\lambda) = u, \qquad \lambda \in \mathbb{C}\backslash F.$$

For every $x \in X_T(\mathbb{C}\backslash F)$, there is a closed set $E \subseteq \mathbb{C}\backslash F$ such that $x \in X_T(E)$; that is, there exists an analytic $X$-valued function $f_x: \mathbb{C}\backslash E \to X$ such that

$$(\lambda - T)f_x(\lambda) = x, \qquad \lambda \in \mathbb{C}\backslash E.$$

We define an analytic complex function on $\mathbb{C}$ as

$$F(\lambda) = \begin{cases} \langle f_x(\lambda), u\rangle, & \lambda \in \mathbb{C}\backslash E, \\ \langle x, f_u^*(\lambda)\rangle, & \lambda \in \mathbb{C}\backslash F.\end{cases}$$

Note that if $\lambda \in (\mathbb{C}\backslash E) \cap (\mathbb{C}\backslash F)$, then

$$\begin{aligned}\langle f_x(\lambda), u\rangle &= \langle f_x(\lambda), (\lambda - T^*)f_u^*(\lambda)\rangle \\ &= \langle(\lambda - T)f_x(\lambda), f_u^*(\lambda)\rangle = \langle x, f_u^*(\lambda)\rangle;\end{aligned}$$

hence, $F(\lambda)$ is well defined. Because $E \subseteq \mathbb{C}\backslash F$, then $(\mathbb{C}\backslash E) \cup (\mathbb{C}\backslash F) = \mathbb{C}$; hence, $F(\lambda)$ is analytic on the whole plane and

$$\begin{aligned}|F(\lambda)| &= |\langle f_x(\lambda), u\rangle| \le \|f_x(\lambda)\|\,\|u\| \\ &= \|(\lambda - T)^{-1}x\|\,\|u\| \to 0\end{aligned}$$

as $|\lambda| \to +\infty$. According to Liouville's theorem, $F(\lambda) \equiv 0$. Hence for $\lambda \in \mathbb{C}\backslash F$,

$$\langle x, f_u^*(\lambda)\rangle = F(\lambda) = 0.$$

Since $Tx \in X_T(\mathbb{C}\backslash F)$,

$$\langle Tx, f_u^*(\lambda)\rangle = 0, \quad \text{for } \lambda \in \mathbb{C}\backslash F.$$

So for $\lambda \in \mathbb{C}\backslash F$,

$$\begin{aligned}\langle x, u\rangle &= \langle x, (\lambda - T^*)f_u^*(\lambda)\rangle = \langle(\lambda - T)x, f_u^*(\lambda)\rangle \\ &= \lambda\langle x, f_u^*(\lambda)\rangle - \langle Tx, f_u^*(\lambda)\rangle = 0,\end{aligned}$$

that is, $u \in X_T(\mathbb{C}\backslash F)^\perp$. The proof is complete.

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