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QUASI-ISOMORPHISM INVARIANTS FOR TWO CLASSES OF FINITE RANK BUTLER GROUPS

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ABSTRACT. A complete set of numerical quasi-isomorphism invariants is given for a class of torsion-free abelian groups containing all groups of the form $\mathcal{G}[\mathcal{A}]$, where $\mathcal{A} = (A_1, \dots, A_n)$ is an n-tuple of subgroups of the additive rationals and $\mathcal{G}[\mathcal{A}]$ is the cokernel of the diagonal embedding $\bigcap A_i \to \bigoplus A_i$. This classification and its dual include, as special cases, earlier classifications of strongly indecomposable groups of the form $\mathcal{G}[\mathcal{A}]$ and their duals.

The purpose of this note is to show that the complete sets of quasi-isomorphism invariants for strongly indecomposable torsion-free abelian groups of the form $\mathcal{G}(\mathcal{A})$ or $\mathcal{G}[\mathcal{A}]$ given in [AV3, AV4] actually classify these groups without the strong indecomposability assumption and, in fact, classify a strictly larger class of groups. Let $\mathcal{A} = (A_1, \dots, A_n)$ be an n-tuple of subgroups of the additive rationals Q. Then $\mathcal{G}(\mathcal{A})$ is the kernel of the summation map $A_1 \oplus \cdots \oplus A_n \to \sum A_i \subseteq Q$ and $\mathcal{G}[\mathcal{A}]$ is the cokernel of the diagonal embedding $\bigcap A_i \to A_1 \oplus \cdots \oplus A_n$. Groups of the form $\mathcal{G}(\mathcal{A})$ are dual to groups of the form $\mathcal{G}[\mathcal{A}]$ via a quasi-homomorphism duality for Butler groups, which is detailed in [AV4]. Thus it suffices to work with just one of these classes. We choose to focus on the $\mathcal{G}[\mathcal{A}]$'s for the relatively minor reason that this class is usually studied in terms of pure subgroups, while the $\mathcal{G}(\mathcal{A})$'s are studied in terms of homomorphic images; the latter seem to be marginally more difficult to handle.

The known invariants for these groups are the ranks of a relatively small collection of subgroups, which we describe following some additional definitions. Our treatment utilizes basic tools developed in [AV1–7] and summarized and refined in [HM]. Fuchs and Metelli [FM] have obtained similar results using different techniques. The paper [AV8] provides a survey of existing literature on the subject.

In contrast to the usual definition, we define a *type* as an isomorphism class of subgroups of Q. A subgroup X of Q can then be identified with the type $\{\alpha X \mid 0 \neq \alpha \in Q\}$ to which it belongs, and we will do this whenever the context leaves no room for confusion. If g is an element of a torsion-free group G, then $\text{type}(g) = \{\alpha \in Q \mid \alpha g \in G\}$. If X and Y are subgroups of Q, we write

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$X \leq Y$ to indicate their relationship as types. Thus $X \subseteq Y$ implies $X \leq Y$, but not conversely. If G is a torsion-free group and X is a subgroup of Q (or a type), denote $G(X) = \{g \in G | \text{type}(g) \geq X\}$; $G[X] = \bigcap\{\ker f | f\colon G \to X\}$, and for $\mathscr{M}$ a set of types, $G(\mathscr{M}) = \sum\{G(X) | X \in \mathscr{M}\}$. In our notation, capital letters always denote groups, while script capitals denote n-tuples or sets of subgroups of Q. The only exception is the symbol $\mathscr{G}$, which is used to denote the formation of a group from an n-tuple, as in $\mathscr{G}[\mathscr{A}]$. Keeping in mind these conventions should help in avoiding the confusion inherent in the established notational use of [ ] and ( ) for our particular groups. For example, $\mathscr{G}(\mathscr{A})$ denotes a group formed from the n-tuple $\mathscr{A}$, while $G(\mathscr{M})$ denotes the subgroup of G defined above.

Denote by Γ the class of all Butler groups quasi-isomorphic to a direct sum of the form $G = \mathscr{G}[\mathscr{D}_1] \oplus \mathscr{G}[\mathscr{D}_2] \oplus \cdots \oplus \mathscr{G}[\mathscr{D}_m]$ for $\mathscr{D}_1, \ldots, \mathscr{D}_m$ tuples of subgroups of Q, such that:

(Γ1) Each $\mathscr{G}[\mathscr{D}_i]$ is strongly indecomposable; and

(Γ2) If $\text{rank}\,\mathscr{G}[\mathscr{D}_i] \geq 2$, then $\sum\{G(A_k) | A_k \in \mathscr{D}_i\}$ is quasi-isomorphic to $\mathscr{G}[\mathscr{D}_i] \oplus C$, with C completely decomposable.

Note that Γ is closed under quasi-summands. We show in Proposition 6 that Γ contains all groups of the form $\mathscr{G}[\mathscr{A}]$. The main theorem of the paper is the following.

Theorem 7. *Let G and H be Butler groups in the class Γ. Then G is quasi-isomorphic to H if and only if* $\text{rank}\, G(\mathscr{M}) = \text{rank}\, H(\mathscr{M})$ *for each set of types $\mathscr{M}$ from the type lattice generated by* $\text{typeset}\, G \cup \text{typeset}\, H$.

This theorem, with the stronger hypothesis that G and H are strongly indecomposable groups of the form $\mathscr{G}[\mathscr{A}]$, appears in various forms in [AV4, FM, HM]. The rest of the paper is devoted to its proof.

An n-tuple $\mathscr{A} = (A_1, \ldots, A_n)$ of subgroups of Q is called *cotrimmed* [Le] provided that for each i, $A_i = A_i + \bigcap_{j \neq i} A_j$. Equivalently, the canonical image of each A_i in $\mathscr{G}[\mathscr{A}]$ is pure. It will be convenient to view $\mathscr{G}[\mathscr{A}]$ as a sum of these pure subgroups. Since multiplying the n-tuple $\mathscr{A}$ by a nonzero rational does not change the isomorphism class of $\mathscr{G}[\mathscr{A}]$ (see [R]), we can assume $1 \in A_i$ and denote by a_i the image in $\mathscr{G}[\mathscr{A}]$ of the element $(0, \ldots, 1, \ldots, 0) \in A_1 \oplus \cdots \oplus A_i \oplus \cdots \oplus A_n$ (that has a 1 in the ith position and 0's elsewhere). Then $\mathscr{G}[\mathscr{A}] = \sum A_i a_i$ and $a_1 + \cdots + a_n = 0$, while any proper subset of the a_i's is linearly independent. Our first two results are well known.

Lemma 1. *Let G be a Butler group and X a type.*

(a) $\text{rank}(G(X) + G[X]) - \text{rank}\, G[X]$ *is the rank of a maximal X-homogeneous completely decomposable quasi-summand of G.*

(b) *If L is the lattice of types generated by* $\text{typeset}\, G$, *then $G[X]$ is the pure subgroup generated by $G(\mathscr{M})$, where $\mathscr{M} = \{Y \in L | Y \not\leq X\}$.*

(c) *With $\mathscr{M}$ as in* (b), $\text{rank}(G(X) + G[X]) - \text{rank}\, G[X] = \text{rank}\, G(\mathscr{M} \cup \{X\}) - \text{rank}\, G(\mathscr{M})$.

Proof. For (a), see [AV1, Corollary 2.2]; Corollary 1.4 of [ARV] contains a version for representations of finite posets. Part (b) is due to Lady [La]. Part (c) follows from (b).

Theorem 2 (see [AV3] or [AV6]). *Let* $\mathscr{A} = (A_1, \dots, A_n)$ *be a cotrimmed n-tuple of subgroups of* Q *and* $G = \mathscr{G}[\mathscr{A}]$. *The following are equivalent.*

(a) G *is strongly indecomposable.*

(b) $\operatorname{rank} G(A_i) = 1$ *for* $1 \le i \le n$.

(c) $\operatorname{End}(G)$ *is isomorphic to a subring of* Q.

An easy corollary of Theorem 2 will be used in the proof of the main theorem.

Corollary 3. *Let* $\mathscr{A} = (A_1, \dots, A_n)$ *be a cotrimmed n-tuple of subgroups of* Q *such that* $G = \mathscr{G}[\mathscr{A}]$ *is strongly indecomposable. If* H *is a torsion-free group such that* $\operatorname{rank} H(\mathscr{M}) = \operatorname{rank} G(\mathscr{M})$ *for each subset* $\mathscr{M}$ *of* $\{A_1, \dots, A_n\}$, *then* G *is quasi-isomorphic to a subgroup of* H.

Proof. Write $G = \sum A_i a_i$ with $a_1 + \cdots + a_n = 0$ in G. Theorem 2 and the hypothesis on ranks imply that there are elements $b_i \in H(A_i)$ such that $b_1 + \cdots + b_n = 0$, but any proper subset of the b_i's is independent. Then $a_i \to b_i$ defines a monic quasi-homomorphism of G into H.

We complete the preliminaries with an additional known result. If $\mathscr{A} = (A_1, \dots, A_n)$ is an n-tuple of subgroups of Q and X is a subgroup of Q, we obtain an equivalence relation on the elements of $\mathscr{A}$ by calling A_i *X-equivalent* to A_j provided $X \not\le A_i + A_j$ and by extending via reflexivity and transitivity. Note that $\{A_i\}$ is always an A_i-equivalence class. The utility of this equivalence notion is indicated by the next theorem. To avoid cumbersome notation, we will frequently treat n-tuples as sets and vice versa. We also use $\doteq$ to denote quasi-isomorphism.

Theorem 4 (see [AV2] or [AV6, Proof of Theorem 2.4]. *Let* $\mathscr{A} = (A_1, \dots, A_n)$ *be an n-tuple of subgroups of* Q *and* $G = \mathscr{G}[\mathscr{A}]$.

(a) *If* X *is any subgroup of* Q, *then* $\operatorname{rank} G(X) + 1$ *is the number of X-equivalence classes in* $\mathscr{A}$.

(b) *If* $\mathscr{A}$ *is cotrimmed and* $\mathscr{E}_0 = \{A_i\}, \mathscr{E}_1, \dots, \mathscr{E}_r$ *are the A_i-equivalence classes in* $\mathscr{A}$, *then* $G \doteq \mathscr{G}[\mathscr{E}_1'] \oplus \cdots \oplus \mathscr{G}[\mathscr{E}_r']$, *where, for each* $1 \le j \le r$, $\mathscr{E}_j' = \mathscr{E}_j \cup \{A_i\}$.

We are now ready to give an explicit description of the quasi-decomposition of $\mathscr{G}[\mathscr{A}]$ into strongly indecomposable summands.

Decomposition Algorithm for $G = \mathscr{G}[A]$. Assume $\mathscr{A} = (A_1, \dots, A_n)$ is an n-tuple of subgroups of Q and $G = \mathscr{G}[\mathscr{A}]$.

Let $\{A_1\} = \mathscr{E}_0, \mathscr{E}_1, \dots, \mathscr{E}_r$ be the A_1-equivalence classes in $\mathscr{A}$. By Theorem 4(b), $\mathscr{G}[\mathscr{A}] = \mathscr{G}[\mathscr{E}_1'] \oplus \cdots \oplus \mathscr{G}[\mathscr{E}_r']$ (up to quasi-isomorphism), where $\mathscr{E}_i' = \mathscr{E}_i \cup \{A_1\}$. Assume $A_2 \in \mathscr{E}_i'$ for some (unique) $1 \le i \le r$. Write $\{A_2\} = \mathscr{D}_0, \mathscr{D}_1, \dots, \mathscr{D}_s$ for the A_2-equivalence classes in $\mathscr{E}_i'$, with $s \ge 1$. Again by Theorem 4(b), $\mathscr{G}[\mathscr{E}_i'] \doteq \mathscr{G}[\mathscr{D}_1'] \oplus \cdots \oplus \mathscr{G}[\mathscr{D}_s']$, where $\mathscr{D}_j' = \mathscr{D}_j \cup \{A_2\}$. This in turn produces a further quasi-decomposition of G. Note that each of the summands $H_i = \mathscr{G}[\mathscr{D}_i']$ satisfies $\operatorname{rank} H_i(A_2) = 1$ by Theorem 4(a). Iterate this procedure with $A_3, \dots, A_n$. At the start of the kth stage we have decomposed G into quasi-summands of the form $\mathscr{G}[\mathscr{B}]$ where each $\mathscr{B}$ is a subtuple of $\mathscr{A}$ and the group A_k belongs to precisely one of these tuples, say $\mathscr{B}_0$. We then further decompose G by decomposing $\mathscr{G}[\mathscr{B}_0]$ via Theorem 4(b), using A_k-equivalence classes. After n iterations, we obtain a quasi-decomposition of $\mathscr{G}[\mathscr{A}]$ with some strong properties, as summarized in the next result.

Proposition 5. *Assume $\mathscr{A} = (A_1, \dots, A_n)$ is a cotrimmed n-tuple of subgroups of Q and $G = \mathscr{G}[\mathscr{A}]$. The decomposition algorithm above produces a quasi-decomposition*

$$G \doteq \mathscr{G}[\mathscr{D}_1] \oplus \cdots \oplus \mathscr{G}[\mathscr{D}_m]$$

with the tuples $\mathscr{D}_h$ satisfying the following properties:

(a) *Each $\mathscr{D}_h$ is a subtuple of $\mathscr{A}$ (not necessarily cotrimmed) with at least two components.*
(b) *If $h \neq i$, then $\mathscr{D}_h \cap \mathscr{D}_i$ is a subset of $\{A_1, \dots, A_n\}$ containing at most one element.*
(c) *If $A_k \in \mathscr{D}_h$ and $G_h = \mathscr{G}[\mathscr{D}_h]$, then* $\operatorname{rank} G_h(A_k) = 1$.
(d) *Each $\mathscr{G}[\mathscr{D}_h]$ is strongly indecomposable.*
(e) *For each $1 \leq h \leq m$ there is a strictly decending chain of tuples*

$$\mathscr{T}_0 = \mathscr{A} \supset \mathscr{T}_1 \supset \cdots \supset \mathscr{T}_{\mu(h)} = \mathscr{D}_h,$$

with $\mathscr{T}_{j+1} = \{A_k\} \cup \mathscr{E}$ for some $A_k \in \mathscr{T}_j$ and some A_k-equivalence class $\mathscr{E} \neq \{A_k\}$ in $\mathscr{T}_j$.

Proof. Properties (a) and (b) follow readily from the construction of the algorithm. Property (c) is a consequence of Theorem 4: when the group A_k is used to decompose the unique tuple to which it belongs at the start of the kth stage, the resulting summands H in which A_k appears have $\operatorname{rank} H(A_k) = 1$, as noted in the description of the algorithm. Subsequent decompositions preserve this condition: each summand H in which A_k appears will have $\operatorname{rank} H(A_k)$ equal to one. Property (d) is a consequence of (c) via Theorem 2. Since the tuples $\mathscr{D}_i$ are not necessarily cotrimmed, to apply Theorem 2 we need to observe that if $(B_1, \dots, B_k)$ is the cotrimmed version of a k-tuple $(A_1, \dots, A_k)$ $(B_i = A_i + \bigcap_{j \neq i} A_j)$, then $A_i \subseteq B_i$. Thus, if $H = \mathscr{G}[A_1, \dots, A_k]$ and $\operatorname{rank} H(A_i) = 1$, then $\operatorname{rank} H(B_i) = 1$ since $H = \mathscr{G}[B_1, \dots, B_k]$ and $H(B_i) \subseteq H(A_i)$. Property (e) is a direct consequence of the design of the algorithm.

Proposition 6. *Let $G = \mathscr{G}[\mathscr{A}]$ for some n-tuple $\mathscr{A}$ of subgroups of Q. Write $G \doteq \mathscr{G}[\mathscr{D}_1] \oplus \cdots \oplus \mathscr{G}[\mathscr{D}_m]$ as in Proposition 5. If $G_h = \mathscr{G}[\mathscr{D}_h]$, then, for $k \neq h$,* rank $\sum\{G_h(A_i) | A_i \in \mathscr{D}_k\}$ *is either 0 or 1. In particular, G belongs to the class* Γ.

Proof. Fix $k \neq h$ between 1 and m and abbreviate $\bigcap \mathscr{D}_h = \bigcap\{A_i | A_i \in \mathscr{D}_h\}$. Using Proposition 5(e), the tuple $\mathscr{D}_h$ is obtained via a sequence of tuples $\mathscr{T}_0 = \mathscr{A}, \mathscr{T}_1, \dots, \mathscr{T}_\mu = \mathscr{D}_h$, with $\mathscr{T}_j$ obtained as a subtuple of $\mathscr{T}_{j-1}$ by taking, for $A_{j(h)} \in \mathscr{T}_{j-1}$, an $A_{j(h)}$-equivalence class in $\mathscr{T}_{j-1}$ and adjoining $A_{j(h)}$. We will abbreviate $A_j = A_{j(h)}$. There is an analogous sequence $\{T_j'\}$ for the tuple $\mathscr{D}_k$. Let $i(1) \geq 1$ be the smallest index i such that $\mathscr{T}_i \neq \mathscr{T}_i'$. The subtuples $\mathscr{D}_h$ and $\mathscr{D}_k$ of $\mathscr{T}_{i(1)-1}$ belong to different $A_{i(1)}$-equivalence classes, since $\mathscr{T}_{i(1)} \neq \mathscr{T}_{i(1)}'$ are formed from different $A_{i(1)}$-equivalence classes as in Proposition 5(e). Thus, $\bigcap \mathscr{D}_h + \bigcap \mathscr{D}_k \geq A_{i(1)}$ by the definition of equivalence. Next let $i(2)$ be the smallest index $i > i(1)$ (if one exists) such that $\mathscr{T}_i$ does not contain $A_{i(1)}$. Then $A_{i(1)}$ and $\mathscr{D}_h$ are in distinct $A_{i(2)}$-equivalence classes in $\mathscr{T}_{i(2)-1}$ (where $A_{i(2)} \in \mathscr{T}_{i(2)-1}$), so that $A_{i(2)} \leq \bigcap \mathscr{D}_h + A_{i(1)}$. Continuing in this way, we obtain an increasing sequence of indices $i(1) < \cdots < i(t) \leq \mu$ such that

$A_{i(j+1)} \le A_{i(j)} + \bigcap \mathscr{D}_h$ and $A_{i(t)}$ belongs to $\mathscr{T}_\mu = \mathscr{D}_h$. Thus, we obtain the chain of inequalities

$$\bigcap \mathscr{D}_h + \bigcap \mathscr{D}_k \ge A_{i(1)} + \bigcap \mathscr{D}_h \ge A_{i(2)} + \bigcap \mathscr{D}_h \ge \cdots \ge A_{i(t)}.$$

Now suppose A_l is an element of $\mathscr{D}_k$. Then

$$\bigcap \mathscr{D}_h + A_l \ge \bigcap \mathscr{D}_h + \bigcap \mathscr{D}_k \ge A_{i(t)}.$$

If $\mathscr{E}_1$ and $\mathscr{E}_2$ are two A_l-equivalence classes in $\mathscr{D}_h$, then $\bigcap \mathscr{E}_1 + \bigcap \mathscr{E}_2 \ge A_l$ by the definition of equivalence. Since $\bigcap \mathscr{E}_1 + \bigcap \mathscr{E}_2 \ge \bigcap \mathscr{D}_h$, it follows that

$$\bigcap \mathscr{E}_1 + \bigcap \mathscr{E}_2 \ge \bigcap \mathscr{D}_h + A_l \ge A_{i(t)}.$$

Thus, every A_l-equivalence class in $\mathscr{D}_h$ is a union of $A_{i(t)}$-equivalence classes. However, by Proposition 5(c) and Theorem 4, there are only two $A_{i(t)}$-equivalence classes in $\mathscr{D}_h$, namely, $\{A_{i(t)}\}$ and $\mathscr{D}_h \backslash \{A_{i(t)}\}$. Consequently, there are at most two A_l-equivalence classes in $\mathscr{D}_h$. Moreover, if there are exactly two A_l-equivalence classes, they must be $\{A_{i(t)}\}$ and $\mathscr{D}_h \backslash \{A_{i(t)}\}$. In this case, by 4(a), $\operatorname{rank} G_h(A_l) = 1$, where $G_h = \mathscr{G}[\mathscr{D}_h]$. Moreover, by the definition of A_l-equivalence, $G_h(A_l)$ is the pure subgroup generated by the image of $A_{i(t)}$ in G_h, namely, $G_h(A_{i(t)})$. As we let A_l range over the elements of $\mathscr{D}_k$, we see that $\sum \{G_h(A_l) | A_l \in \mathscr{D}_k\} \subseteq G_h(A_{i(t)})$. Since the group $G_h(A_{i(t)})$ has rank one by 5(c), the proof of the first assertion of the proposition is complete. To verify the assertion that G belongs to Γ, note that condition $(\Gamma 1)$ holds for G by Proposition 5(d) and that condition $(\Gamma 2)$ is a direct consequence of the first part of Proposition 6.

By Proposition 6, Γ contains all Butler groups quasi-isomorphic to groups of the form $\mathscr{G}[\mathscr{A}]$; however, there are many groups in Γ which are not quasi-isomorphic to a group of the form $\mathscr{G}[\mathscr{A}]$. For example, if $p_1, \ldots, p_6$ are distinct primes and A_i is the smallest subring of Q containing p_i^{-1}, then the group $G = \mathscr{G}[A_1, A_2, A_3] \oplus \mathscr{G}[A_4, A_5, A_6]$ is not quasi-isomorphic to a $\mathscr{G}[\mathscr{A}]$ (see [FM, Example 2.5]). But G belongs to Γ because, for instance, $\operatorname{Hom}(A_i, \mathscr{G}[A_4, A_5, A_6]) = 0$ if $1 \le i \le 3$, so that $G(A_1) + G(A_2) + G(A_3) = \mathscr{G}[A_1, A_2, A_3]$ and $(\Gamma 2)$ is satisfied.

We are ready for the main theorem.

Theorem 7. *Let G and H be Butler groups belonging to the class Γ. Then G is quasi-isomorphic to H if and only if* $\operatorname{rank} G(\mathscr{M}) = \operatorname{rank} H(\mathscr{M})$ *for each set of types $\mathscr{M}$ from the type lattice generated by* typeset $G \cup$ typeset H.

Proof. The only if direction is clear. For the converse, by definition of Γ, we have $G \doteq \mathscr{G}[\mathscr{D}_1] \oplus \cdots \oplus \mathscr{G}[\mathscr{D}_m]$ with the $\mathscr{D}_i$ tuples of subgroups of Q such that $(\Gamma 1)$ and $(\Gamma 2)$ are satisfied. It is easy to check that, without loss of generality, we may take the $\mathscr{D}_i$'s to be cotrimmed. In this case, the lattice generated by typeset G is the same as the lattice generated by the entries of the $\mathscr{D}_i$'s ([Le] or [FM]). Abbreviate $G_i = \mathscr{G}[\mathscr{D}_i]$. By $(\Gamma 1)$ each G_i is strongly indecomposable. Let C be the direct sum of the G_i's which have rank one. By Lemma 1(c) and the hypotheses, for each type X we have

$$\operatorname{rank}(G(X) + G[X]) - \operatorname{rank} G[X] = \operatorname{rank}(H(X) + H[X]) - \operatorname{rank} H[X].$$

It follows from Lemma 1(a) that C is quasi-isomorphic to the direct sum of the rank one summands in a quasi-decomposition of H into strongly indecomposable groups. As a consequence we may write $G \doteq G' \oplus C$ and $H \doteq H' \oplus C$ where G' and H' are again in Γ and have no rank one quasi-summands. In addition, since $\operatorname{rank} G(\mathscr{M}) = \operatorname{rank} G'(\mathscr{M}) + \operatorname{rank} C(\mathscr{M})$ for each set of types $\mathscr{M}$, the groups G' and H' inherit the hypotheses of the theorem. Thus we may reduce to the case where G and H have no rank one quasi-summands. In particular, $\operatorname{rank} \mathscr{G}[\mathscr{D}_i] \geq 2$ for each i, so that the quasi-isomorphism in $(\Gamma 2)$ holds for each i.

Write $\mathscr{D}_1 = (A_1, \ldots, A_k)$ and denote $G' = G(\mathscr{D}_1) = \sum_{i=1}^k G(A_i)$ and $H' = H(\mathscr{D}_1) = \sum_{i=1}^k H(A_i)$. By hypothesis, the ranks of G' and H' are equal. Moreover, by $(\Gamma 2)$, $G' \doteq G_1 \oplus C$, where C is a completely decomposable group. We will show that $H' \doteq H_1 \oplus C$ for some subgroup H_1 of H'. Let X be the type of a nonzero rank one summand of C. The group G' is a homomorphic image of $\bigoplus_{i=1}^k G(A_i)$, so for some $1 \leq i \leq k$ there is a nonzero map from $G(A_i)$ to X. For convenience take $i = 1$. Then $X \geq A_1$, so that $G'(X) \subseteq G(X) = G(A_1) \cap G(X) \subseteq G'(X)$ and $G(X) = G'(X)$. Similarly, $H(X) = H'(X)$. Clearly, $G'[X]$ contains the pure subgroup G'' generated by $\sum_{i=1}^k G(A_i)[X]$. Moreover, G'/G'' is a homomorphic image of $\bigoplus_{i=1}^k G(A_i)/(G(A_i)[X])$ and, hence, has outer type less than or equal to X. It follows that $G'[X] = G''$. Similarly, $H'[X]$ is purely generated by $\sum_{i=1}^k H(A_i)[X]$. By Lemma 1(b), each $G(A_i)[X]$ and $H(A_i)[X]$ is purely generated by $G(\mathscr{M}_i)$ for $\mathscr{M}_i$ a set of types T satisfying $A_i \leq T \not\leq X$. If $\mathscr{M} = \bigcup \mathscr{M}_i$ then $G'[X]$ is the pure subgroup generated by $G(\mathscr{M})$ and $H'[X]$ is the pure subgroup generated by $H(\mathscr{M})$. We have established the following equalities:

$$\operatorname{rank} G'(X) = \operatorname{rank} G(X) = \operatorname{rank} H(X) = \operatorname{rank} H'(X);$$
$$\operatorname{rank} G'[X] = \operatorname{rank} G(\mathscr{M}) = \operatorname{rank} H(\mathscr{M}) = \operatorname{rank} H'[X];$$
$$\begin{aligned}\operatorname{rank}(G'(X) + G'[X]) &= \operatorname{rank} G(\mathscr{M} \cup \{X\}) = \operatorname{rank} H(\mathscr{M} \cup \{X\}) \\ &= \operatorname{rank}(H'(X) + H'[X]).\end{aligned}$$

By Lemma 1(a), $\operatorname{rank} G(\mathscr{M} \cup \{X\}) - \operatorname{rank} G(\mathscr{M}) = \operatorname{rank} H(\mathscr{M} \cup \{X\}) - \operatorname{rank} H(\mathscr{M})$ is the rank of a maximal X-homogeneous completely decomposable quasi-summand of G' and of H'. It follows that $H' \doteq H_1 \oplus C$, where C is a completely decomposable group such that $G' \doteq G_1 \oplus C$.

The next step is to show that G_1 embeds into H_1. Note that for any subset $\mathscr{M}$ of $\{A_1, \ldots, A_k\}$, $\operatorname{rank} G_1(\mathscr{M}) = \operatorname{rank} H_1(\mathscr{M})$, since $\operatorname{rank} G'(\mathscr{M}) = \operatorname{rank} G(\mathscr{M}) = \operatorname{rank} H(\mathscr{M}) = \operatorname{rank} H'(\mathscr{M})$ and $\operatorname{rank} G'(\mathscr{M}) = \operatorname{rank} G_1(\mathscr{M}) + \operatorname{rank} C(\mathscr{M})$. Therefore, G_1 is quasi-isomorphic to a subgroup of H_1 by Corollary 3. Similarly, each $G_i = \mathscr{G}[\mathscr{D}_i]$ is quasi-isomorphic to a subgroup of H. By symmetry, any strongly indecomposable quasi-summand of $H \in \Gamma$ is quasi-isomorphic to a subgroup of G. It follows that there is a nonzero map $G_i \to H \to G$. The image of this map has nonzero projection onto some G_j. By the same reasoning, we can then obtain a nonzero composition

$$G_i \to H \to G \to G_j \to H \to G \to G_k$$

for some k. If we continue this process, eventually one of the subscripts on the G's will repeat. At this point, for some index $\imath$, we will have a nonzero composition $G_\iota \to H \to G_\iota$ that is a quasi-automorphism of G_ι, since G_ι strongly indecomposable implies $\mathrm{End}(G_\iota) \subset Q$ by Theorem 2. As a consequence, we may write $G \doteq G_\iota \oplus G'$ and $H \doteq G_\iota \oplus H'$ for some groups G' and H'; however, the class Γ is closed under quasi-summands, so that G' and H' belong to Γ. As noted previously, the quasi-direct decompositions $G \doteq G_\iota \oplus G'$ and $H \doteq G_\iota \oplus H'$, along with the hypothesis of the theorem, imply that $\operatorname{rank} G'(\mathscr{M}) = \operatorname{rank} H'(\mathscr{M})$ for each set of types $\mathscr{M}$ from the lattice generated by $\operatorname{typeset} G' \cup \operatorname{typeset} H' \subseteq \operatorname{typeset} G \cup \operatorname{typeset} H$. The confluence of these remarks allows us to apply an induction on rank to G' and H', and the proof is complete.

Let Γ' be the class of all Butler groups quasi-isomorphic to groups of the form $G = \mathscr{G}(\mathscr{D}_1) \oplus \cdots \oplus \mathscr{G}(\mathscr{D}_m)$, where each $\mathscr{D}_i$ is a tuple of subgroups of Q, such that each $\mathscr{G}(\mathscr{D}_i)$ is strongly indecomposable; and if $\operatorname{rank} \mathscr{G}(\mathscr{D}_i) \geq 2$, then $G/\bigcap\{G[A_k] | A_k \in \mathscr{D}_i\} \doteq \mathscr{G}(\mathscr{D}_i) \oplus C$, with C completely decomposable. Applying the duality of [AV4] immediately provides the following.

Corollary 8. *Let G and H be Butler groups in the class Γ'. Then G and H are quasi-isomorphic if and only if* $\operatorname{rank}(\bigcap_{X\in\mathscr{M}} G[X]) = \operatorname{rank}(\bigcap_{X\in\mathscr{M}} H[X])$ *for each subset $\mathscr{M}$ of the lattice of types generated by* $\operatorname{typeset} G \cup \operatorname{typeset} H$.

Remark. Analogs of Theorem 7 and Corollary 8 hold in the context of representations of finite posets, a fact we note but do not prove. The interested reader can make the minor changes needed to obtain the (more general) proofs by referring to [AV6].

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