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A TRANSITIVITY THEOREM FOR ALGEBRAS OF ELEMENTARY OPERATORS

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ABSTRACT. Let $\mathscr{A}$ be a C^*-algebra and $\mathscr{E}$ the algebra of all elementary operators on $\mathscr{A}$, and let $\vec{a} = (a_1, \dots, a_n)$, $\vec{b} = (b_1, \dots, b_n) \in \mathscr{A}^n$. It is proved that $\vec{b}$ is contained in the closure of the set $\{(Ea_1, \dots, Ea_n) : E \in \mathscr{E}\}$ if and only if each complex linear combination $\sum_{j=1}^n \lambda_j b_j$ is contained in the closed two-sided ideal generated by $\sum_{j=1}^n \lambda_j a_j$. In particular, a bounded linear operator on $\mathscr{A}$ preserves all closed two-sided ideals if and only if it is in the strong closure of $\mathscr{E}$.

1. Introduction, motivation, and notation

An *elementary operator* on a ring $\mathscr{A}$ is a map $E : \mathscr{A} \to \mathscr{A}$ of the form

$$Ex = \sum_{i=1}^{m} u_i x v_i \qquad (x \in \mathscr{A}),$$

where $\vec{u} = (u_1, \dots, u_m)$ and $\vec{v} = (v_1, \dots, v_m)$ are fixed m-tuples of elements of $\mathscr{A}$. In the last decade there has been considerable interest in such operators, especially in the cases when $\mathscr{A}$ is the algebra $\mathscr{B}(\mathscr{H})$ of all bounded operators on a Hilbert space $\mathscr{H}$, the Calkin algebra or a general prime C^*-algebra (see [1, 7, 8] and their bibliographies). In this note we will be concerned with algebras of elementary operators on general C^*-algebras.

Our study here is motivated by the following classical algebraic considerations. For any unital algebra $\mathscr{A}$ (over some field) the set $\mathscr{E}$ of all elementary operators on $\mathscr{A}$ is again an algebra (with the usual operations of addition, multiplication by scalars, and composition of operators). The algebra $\mathscr{A}$ itself can be regarded as a (left) module over $\mathscr{E}$, the submodules of which are precisely the two-sided ideals of $\mathscr{A}$, and the module endomorphisms of $\mathscr{A}$ are just the multiplications by elements of the centre $\mathscr{C}$ of $\mathscr{A}$. For any positive integer n the direct sum $\mathscr{A}^n$ of n copies of $\mathscr{A}$ is then, of course, also an $\mathscr{E}$-module. Let us consider the following question.

Given $\vec{a}$, $\vec{b} \in \mathscr{A}^n$, under what conditions does there exist an elementary operator $E \in \mathscr{E}$ such that $E\vec{a} = \vec{b}$ (that is, $Ea_j = b_j$ for $j = 1, \dots, n$)?

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One necessary condition is obvious: since $Ea_j = b_j$ $(j = 1, \dots, n)$ implies that $E(\sum_{j=1}^n \lambda_j a_j) = \sum_{j=1}^n \lambda_j b_j$ for arbitrary central elements $\lambda_j \in \mathscr{A}$, we see that $\sum_{j=1}^n \lambda_j b_j$ must be in the two-sided ideal generated by $\sum_{j=1}^n \lambda_j a_j$. To shorten the notation, for every $\vec{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathscr{C}^n$ and every $\vec{x} = (x_1, \dots, x_n)$ the linear combination $\sum_{j=1}^n \lambda_j x_j$ will be denoted by $\vec{\lambda} \cdot \vec{x}$. Also, let us agree that the word 'ideal' means a two-sided ideal and, for each $x \in \mathscr{A}$, denote by $\langle x \rangle$ the ideal generated by x. We may now ask the following

Question 1. *Is the condition that $\vec{\lambda} \cdot \vec{b} \in \langle \vec{\lambda} \cdot \vec{a} \rangle$ for each $\vec{\lambda} \in \mathscr{C}^n$ also sufficient for the existence of an elementary operator E on $\mathscr{A}$ such that $E\vec{a} = \vec{b}$?*

In the special case, when $\mathscr{A}$ has no nontrivial ideals, the answer to the last question is affirmative. Namely, in this case the condition reduces to the requirement that $\vec{\lambda} \cdot \vec{a} = 0$ implies $\vec{\lambda} \cdot \vec{b} = 0$ (for each $\vec{\lambda} \in \mathscr{C}^n$), and since $\mathscr{A}$ is a simple $\mathscr{E}$-module, the existence of $E \in \mathscr{E}$ satisfying $E\vec{a} = \vec{b}$ follows from the Jacobson density theorem [11, p. 220]. In general, however, the answer to Question 1 is negative, as is shown by the following example.

Example. Let $\mathscr{H}$ be an infinite-dimensional (complex) vector space, $\mathscr{L}$ the algebra of all linear operators on $\mathscr{H}$, $\mathscr{F}$ the ideal of all finite rank operators, and $c \in \mathscr{L}$ a fixed operator such that $c \notin \mathscr{F}$ and $c^2 \in \mathscr{F}$. Let $\mathscr{A}$ be the subalgebra of $\mathscr{L}$ generated by $\mathscr{F}$, c, and the identity operator 1, and put $\vec{a} = (1, c)$, $\vec{b} = (1, c^2)$. Then clearly the centre of $\mathscr{A}$ consists of scalars only, and it is easy to verify that the only proper ideals in $\mathscr{A}$ are 0, $\mathscr{F}$, and the ideal generated by $\mathscr{F} \cup \{c\}$. It follows that, with $\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathscr{C}^2$, the ideal $\langle \vec{\lambda} \cdot \vec{a} \rangle = \langle \lambda_1 + \lambda_2 c \rangle$ is proper only if $\lambda_1 = 0$, and the condition $\vec{\lambda} \cdot \vec{b} \subset \langle \vec{\lambda} \cdot \vec{a} \rangle$ is satisfied for all $\vec{\lambda} \in \mathscr{C}^2$. On the other hand, we shall now see that there is no elementary operator E on $\mathscr{A}$ such that $E\vec{a} = \vec{b}$.

Assume to the contrary, that there exist u_i, $v_i \in \mathscr{A}$ $(i = 1, \dots, m)$ such that

$$\sum_{i=1}^m u_i 1 v_i = 1 \quad \text{and} \quad \sum_{i=1}^m u_i c v_i = c^2.$$

Since the quotient algebra $\mathscr{A}/\mathscr{F}$ is obviously commutative, the last two identities imply that $c^2 = c \pmod{\mathscr{F}}$, but this is in contradiction with the fact that $c \notin \mathscr{F}$, $c^2 \in \mathscr{F}$.

We are now going to say a few words about the analogy of the above purely algebraic question in the context of Banach algebras. Let $\mathscr{A}$ be a complex unital Banach algebra, and denote by $[x]$ the closed ideal generated by an element $x \in \mathscr{A}$. Given a positive integer n and $\vec{a}$, $\vec{b} \in \mathscr{A}^n$ we may now ask

Question 2. *When does there exist a sequence of elementary operators E_k $(k = 1, 2, \dots)$ on $\mathscr{A}$ such that $E_k\vec{a}$ converge to $\vec{b}$?*

Obviously a necessary condition for the existence of such a sequence of elementary operators is that $\vec{\lambda} \cdot \vec{b} \in [\vec{\lambda} \cdot \vec{a}]$ for each $\vec{\lambda} \in \mathscr{C}^n$, but a simple modification of the example shows that this condition is not sufficient. (Namely, in the Example we replace $\mathscr{L}$ by the algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on a separable Hilbert space $\mathscr{H}$, and $\mathscr{F}$ by the ideal $\mathscr{K}(\mathscr{H})$ of all compact operators; we choose $c \in \mathscr{B}(\mathscr{H})\backslash\mathscr{K}(\mathscr{H})$ so that $c^2 \in \mathscr{K}(\mathscr{H})$, and let $\mathscr{A}$ be the Banach algebra generated by $\mathscr{K}(\mathscr{H})$, c, and the identity operator.) Concerning general Banach algebras, we state here only one simple result.

Proposition 1.1. *Let $\mathscr{A}$ be a complex unital Banach algebra, and assume that $\vec{a} = (a_1, \dots, a_n) \in \mathscr{A}^n$ is such that $[\vec{\lambda} \cdot \vec{a}] = \mathscr{A}$ for every $\vec{\lambda} \neq 0$, $\vec{\lambda} \in \mathbb{C}^n$. Then for every $\vec{b} = (b_1, \dots, b_n) \in \mathscr{A}^n$ there exists an elementary operator E on $\mathscr{A}$ such that $E\vec{a} = \vec{b}$.*

Proof. The proof is by an induction on n. The case $n = 1$ is trivial, so let $n > 1$. We must prove that $\mathscr{E}\vec{a} = \mathscr{A}^n$, and for this it suffices to prove that there exists $F_n = F \in \mathscr{E}$ satisfying

$$Fa_j = 0 \text{ for } j = 1, \dots, n-1 \quad \text{and} \quad \langle Fa_n \rangle = \mathscr{A}. \tag{1}$$

For then, by the same argument, there exists for each $i = 1, \dots, n$ an $F_i \in \mathscr{E}$ satisfying $F_i a_j = 0$ for $j \neq i$ and $\langle F_i a_i \rangle = \mathscr{A}$, and choosing $E_i \in \mathscr{E}$ so that $E_i F_i a_i = b_i$ (which is possible, since $\langle F_i a_i \rangle = \mathscr{A}$), we see that the operator $E \stackrel{\text{def}}{=} \sum_{i=1}^n E_i F_i$ then satisfies $E\vec{a} = \vec{b}$. To prove the existence of F satisfying (1), suppose, to the contrary, that no such F exists, and denote by $\mathscr{N}$ the left ideal in $\mathscr{E}$ consisting of all $F \in \mathscr{E}$ that satisfy $F(a_1, \dots, a_{n-1}) = 0$. Note that $\mathscr{N} a_n$ is an $\mathscr{E}$-submodule of $\mathscr{A}$ and hence, an ideal in $\mathscr{A}$. Since by assumption no $F \in \mathscr{E}$ satisfies (1), $1 \notin \mathscr{N} a_n$; therefore, $\mathscr{N} a_n$ is contained in some proper maximal ideal $\mathscr{M}$ of $\mathscr{A}$. By the induction hypothesis we have $\mathscr{E}(a_1, \dots, a_{n-1}) = \mathscr{A}^{n-1}$. Since $\mathscr{N} a_n \subseteq \mathscr{M}$, the map

$$\varphi : \mathscr{A}^{n-1} \to \mathscr{A}/\mathscr{M}, \qquad \varphi(Fa_1, \dots, Fa_{n-1}) \stackrel{\text{def}}{=} Fa_n + \mathscr{M} \quad (F \in \mathscr{E})$$

is a well-defined homomorphism of $\mathscr{E}$-modules. Let $\varphi_1, \dots, \varphi_{n-1}$ be the components of φ (that is, $\varphi_i : \mathscr{A} \to \mathscr{A}/\mathscr{M}$ are $\mathscr{E}$-module homomorphisms such that $\varphi(x_1, \dots, x_{n-1}) = \sum_{j=1}^{n-1} \varphi_j(x_j)$ for each $(x_1, \dots, x_{n-1}) \in \mathscr{A}^{n-1}$). For each j the kernel of φ_j must contain $\mathscr{M}$ (since for each $m \in \mathscr{M}$ we have $\varphi_j(m) = \varphi_j(m1) = m\varphi_j(1) = 0$ in $\mathscr{A}/\mathscr{M}$). Hence φ_j induces an endomorphism λ_j of the $\mathscr{E}$-module $\mathscr{A}/\mathscr{M}$. Such an endomorphism is necessarily a multiplication by a central element of $\mathscr{A}/\mathscr{M}$, but the centre of each simple algebra is a field and the only field among the complex Banach algebras is the field $\mathbb{C}$ of all complex numbers. It follows that $\lambda_j \in \mathbb{C}$ for each j. By the definition of φ we now have

$$a_n + \mathscr{M} = \varphi(a_1, \dots, a_{n-1}) = \sum_{j=1}^{n-1} \lambda_j a_j + \mathscr{M}.$$

Hence $a_n - \sum_{j=1}^{n-1} \lambda_j a_j \in \mathscr{M}$, but this contradicts the assumption that $[\vec{\lambda} \cdot \vec{a}] = \mathscr{A}$ for each nonzero $\vec{\lambda} \in \mathbb{C}^n$. $\square$

In the remaining part of this note we confine our attention to C^*-algebras, where the necessary condition that $\vec{\lambda} \cdot \vec{b} \in [\vec{\lambda} \cdot \vec{a}]$ for each $\vec{\lambda} \in \mathbb{C}^n$ turns out to be sufficient also for $\vec{b}$ to be in the norm closure of the set $\mathscr{E}\vec{a}$. At the same time the analogous question in the context of von Neumann algebras is answered.

At the end we shall also consider the so-called range inclusion problem for elementary operators on factors.

2. The case of C^*-algebras

Theorem 2.1. *Let $\mathscr{A}$ be a C^*-algebra, $\mathscr{E}$ the algebra of all elementary operators on $\mathscr{A}$, n a positive integer, and $\vec{a}$, $\vec{b} \in \mathscr{A}^n$. Then $\vec{b}$ belongs to the norm closure $\overline{\overline{\mathscr{E}\vec{a}}}$ of $\mathscr{E}\vec{a}$ if and only if for each $\vec{\lambda} \in \mathbb{C}^n$ the element $\vec{\lambda} \cdot \vec{b}$ is contained in the closed ideal $[\vec{\lambda} \cdot \vec{a}]$ generated by $\vec{\lambda} \cdot \vec{a}$.*

If $\mathscr{A}$ is a von Neumann algebra, then $\vec{b}$ belongs to the closure $\overline{\mathscr{E}\vec{a}}$ of $\mathscr{E}\vec{a}$ in the weak operator topology of $\mathscr{A}^n$ if and only if for each $\vec{\lambda} \in \mathscr{C}^n$ (where $\mathscr{C}$ is the centre of $\mathscr{A}$) the relation $\vec{\lambda} \cdot \vec{a} = 0$ implies $\vec{\lambda} \cdot \vec{b} = 0$.

For a von Neumann algebra $\mathscr{R} \subseteq \mathscr{B}(\mathscr{H})$ we denote by $M_{m,n}(\mathscr{R})$ the set of all $m \times n$ matrices with entries from $\mathscr{R}$, and we let $M_n(\mathscr{R}) = M_{n,n}(\mathscr{R})$. Further, we identify $\mathscr{R}^n$ with $M_{n,1}(\mathscr{R})$ so that elements of $\mathscr{R}^n$ are regarded as operators from $\mathscr{H}$ to $\mathscr{H}^n$. To prove Theorem 2.1 we first need a generalization of the well-known fact that every weakly closed ideal in a von Neumann algebra is generated by a central projection [4, p. 443].

Lemma 2.2. *Let $\mathscr{R}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Then for each right $\mathscr{R}$-submodule $\mathscr{M}$ in $\mathscr{R}^n$ which is closed in the weak operator topology there exists an idempotent $p \in M_n(\mathscr{R})$ such that $\mathscr{M} = p\mathscr{R}^n$. Moreover, if $\mathscr{M}$ is an $\mathscr{R}$-bimodule, then $p \in M_n(\mathscr{C})$, where $\mathscr{C}$ is the centre of $\mathscr{R}$.*

Proof. Since $\mathscr{M}$ is a right $\mathscr{R}$-module, we have

$$\mathscr{M} = \mathscr{M}\mathscr{R} = \mathscr{M} M_{1,n}(\mathscr{R}) M_{n,1}(\mathscr{R}).$$

Observe that $\mathscr{M} M_{1,n}(\mathscr{R})$ is a right ideal in the von Neumann algebra $M_n(\mathscr{R})$. Hence there exists a projection $p \in M_n(\mathscr{R})$ such that $\overline{\mathscr{M} M_{1,n}(\mathscr{R})} = pM_n(\mathscr{R})$. It follows that

$$\mathscr{M} = \mathscr{M} M_{1,n}(\mathscr{R}) M_{n,1}(\mathscr{R}) \subseteq pM_n(\mathscr{R}) M_{n,1}(\mathscr{R}) = pM_{n,1}(\mathscr{R}).$$

Since $\mathscr{M}$ is closed in the weak operator topology, the reverse inclusion also holds:

$$\mathscr{M} = \overline{\mathscr{M} M_{1,n}(\mathscr{R}) M_{n,1}(\mathscr{R})} \supseteq \overline{\mathscr{M} M_{1,n}(\mathscr{R})} M_{n,1}(\mathscr{R}) = pM_n(\mathscr{R}) M_{n,1}(\mathscr{R}).$$

Hence $\mathscr{M} = pM_{n,1}(\mathscr{R}) = p\mathscr{R}^n$.

Finally, suppose that $\mathscr{M}$ is an $\mathscr{R}$-bimodule. We have just proved that $\mathscr{M} = p\mathscr{R}^n$ for some projection $p \in M_n(\mathscr{R})$, and we must now show that $p \in M_n(\mathscr{C})$. Denote by $\mathscr{R}^{(n)}$ the subalgebra of $M_n(\mathscr{R})$ consisting of all diagonal matrices with the same element along the diagonal. Since $\mathscr{M}$ is a left $\mathscr{R}$-module, we have $\mathscr{R}^{(n)}\mathscr{M} \subseteq \mathscr{M}$ or (equivalently) $(1-p)\mathscr{R}^{(n)}p\mathscr{R}^n = 0$. This implies that $(1-p)\mathscr{R}^{(n)}p = 0$; hence, p commutes with the von Neumann algebra $\mathscr{R}^{(n)}$. It is well known (and easy to verify) that the commutant of $\mathscr{R}^{(n)}$ is $M_n(\mathscr{R}')$ (where $\mathscr{R}'$ is the commutant of $\mathscr{R}$); hence, $p \in M_n(\mathscr{R}) \cap M_n(\mathscr{R}') = M_n(\mathscr{R} \cap \mathscr{R}') = M_n(\mathscr{C})$. □

Proof of Theorem 2.1. We may identify the C^*-algebra $\mathscr{A}$ with its image under the universal representation on some Hilbert space $\mathscr{H}$. It is well known, then, that each bounded linear functional on $\mathscr{A}$ can be uniquely extended to a weak-operator continuous linear functional on the weak-operator closure $\overline{\mathscr{A}}$ of $\mathscr{A}$ and, consequently, that $\overline{\overline{\mathscr{K}}} = \overline{\mathscr{K}} \cap \mathscr{A}$ for each convex subset $\mathscr{K}$ of $\mathscr{A}$ (where one bar denotes the closure in the weak operator topology and two bars denote the norm closure; see [4, p. 713]). It is clear that (if n is finite) these properties hold also for $\mathscr{A}^n$ (in place of $\mathscr{A}$); in particular, for each $\vec{a} \in \mathscr{A}^n$ we have $\overline{\overline{\mathscr{E}\vec{a}}} = \mathscr{A}^n \cap \overline{\mathscr{E}\vec{a}}$. Let $\vec{b} \in \mathscr{A}^n$ be such that $\vec{b} \notin \overline{\overline{\mathscr{E}\vec{a}}}$. Then $\vec{b} \notin \overline{\mathscr{E}\vec{a}}$. Since $\overline{\mathscr{E}\vec{a}}$ is an $\overline{\mathscr{A}}$-bimodule in $\overline{\mathscr{A}}^n$, by Lemma 2.2 there exists a projection $p \in M_n(\mathscr{C})$ (where $\mathscr{C}$ is the centre of $\overline{\mathscr{A}}$) such that $\overline{\mathscr{E}\vec{a}} = p\overline{\mathscr{A}^n}$. With $q = 1 - p$, we now have $q\vec{a} = 0$ and $q\vec{b} \neq 0$ (since $\vec{b} \notin \overline{\mathscr{E}\vec{a}}$). Thus, if c_j $(j = 1, \ldots, n)$ are the entries of a suitable row of q, then

$$\sum_{j=1}^{n} c_j a_j = 0, \qquad \sum_{j=1}^{n} c_j b_j \neq 0, \tag{2}$$

and $c_j \in \mathscr{C}$ for each j. Now choose any irreducible representation π of $\overline{\mathscr{A}}$ such that $\pi(\sum_{j=1}^{n} c_j b_j) \neq 0$ (this is possible by elementary C^*-theory [10, p. 147]). Since the centre of each irreducible algebra consists only of scalar multiples of the identity we have $\pi(c_j) = \lambda_j 1$, where $\lambda_j \in \mathbb{C}$. Hence relations (2) imply

$$\sum_{j=1}^{n} \lambda_j \pi(a_j) = 0 \quad \text{and} \quad \sum_{j=1}^{n} \lambda_j \pi(b_j) \neq 0$$

or

$$\sum_{j=1}^{n} \lambda_j a_j \in \ker \pi \cap \mathscr{A} \quad \text{and} \quad \sum_{j=1}^{n} \lambda_j b_j \notin \ker \pi \cap \mathscr{A}.$$

Thus, with $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$, we have $\vec{\lambda} \cdot \vec{b} \notin [\vec{\lambda} \cdot \vec{a}]$. This proves the nontrivial part of the statement in the theorem concerning C^*-algebras.

The statement concerning von Neumann algebras follows immediately from Lemma 2.2: if $\vec{b} \notin \overline{\mathscr{E}\vec{a}}$, then there exists a projection $q \in M_n(\mathscr{C})$ such that $q\vec{a} = 0$ and $q\vec{b} \neq 0$; hence, for some row $\vec{\lambda}$ of q we have $\vec{\lambda} \cdot \vec{a} = 0$ and $\vec{\lambda} \cdot \vec{b} \neq 0$. □

The following result is an obvious consequence of Theorem 2.1.

Corollary 2.3. *Let $\mathscr{A}$ be a C^*-algebra and $\mathscr{B}(\mathscr{A})$ the algebra of all bounded linear operators on $\mathscr{A}$. Then the closure in the strong operator topology (= point-norm topology) of the algebra $\mathscr{E}$ of all elementary operators on $\mathscr{A}$ (as a subset of $\mathscr{B}(\mathscr{A})$) consists precisely of operators $\varphi \in \mathscr{B}(\mathscr{A})$ that satisfy $\varphi(\mathscr{K}) \subseteq \mathscr{K}$ for each closed ideal $\mathscr{K}$ in $\mathscr{A}$.*

If $\mathscr{A}$ is a von Neumann algebra, then $\mathscr{E}$ is a dense subset of the space of all bounded module endomorphisms of $\mathscr{A}$ over the centre $\mathscr{C}$ equipped with the point-weak operator topology. (Here the point-weak operator topology on $\mathscr{B}(\mathscr{A})$ is defined by the family of seminorms $\psi \to |\langle \psi(a)\xi, \eta\rangle|$, where ξ and η are arbitrary vectors from the Hilbert space on which $\mathscr{A}$ is acting.)

An operator $\varphi \in \mathscr{B}(\mathscr{A})$ is called a *local elementary operator* if for each $x \in \mathscr{A}$ there exists an elementary operator E (depending on x) such that

$\varphi x = Ex$. Larson and Sourour observed in [5] that for each infinite-dimensional Banach space $\mathscr{X}$ there exist on $\mathscr{B}(\mathscr{X})$ nonelementary local elementary operators. Corollary 2.3 implies that on C^*-algebras local elementary operators can be strongly approximated by elementary operators.

To study the range inclusion problem for elementary operators, we shall need a sharper form of Proposition 1.1 for factors, but to prove it, we need a lemma.

Lemma 2.4. *Let $\mathscr{A}$ be a unital prime C^*-algebra (or, more generally, a unital complex ultraprime Banach algebra in the sense [9]), $\mathscr{E}$ the algebra of all elementary operators on $\mathscr{A}$, $\mathscr{S}$ a finite-dimensional subspace of $\mathscr{A}$, and $b \in \mathscr{A}\backslash\mathscr{S}$. Then there exists $E \in \mathscr{E}$ such that $E\mathscr{S} = 0$ and $Eb \neq 0$.*

Proof. The proof is by an induction on the dimension n of $\mathscr{S}$. First assume that $n = 1$, and choose a nonzero element $a \in \mathscr{S}$. If for each $E \in \mathscr{E}$ the condition $Ea = 0$ implies $Eb = 0$, then the correspondence $Ea \to Eb$ is a well-defined $\mathscr{A}$-bimodule homomorphism from the ideal $\mathscr{E}a$ to $\mathscr{A}$, which maps a to b. By [7, Proposition 2.5] (or, if $\mathscr{A}$ is a general ultraprime Banach algebra, by [9, Theorem 4.1]) each such homomorphism is necessarily a multiplication by certain $\gamma \in \mathbb{C}$. Hence we have $b = \gamma a$; but this is in contradiction with $b \notin \mathscr{S}$. Hence there exists an $E \in \mathscr{E}$ such that $Ea = 0$ and $Eb \neq 0$.

Now let n be any positive integer, assume inductively that the lemma holds for all subspaces of dimension at most n, and let $\mathscr{S}$ be an arbitrary $(n+1)$-dimensional subspace of $\mathscr{A}$. Choose a basis $\{a_1, \dots, a_n, a\}$ for $\mathscr{S}$ and denote by $\mathscr{T}$ the span of $\{a_1, \dots, a_n\}$. By the inductive hypothesis there exists $F \in \mathscr{E}$ such that $F\mathscr{T} = 0$ and $Fb \neq 0$. If $Fb \notin \mathbb{C}Fa$, then (by the already proved case $n = 1$) there exists $G \in \mathscr{E}$ such that $GFa = 0$ and $GFb \neq 0$; hence, $E \stackrel{\text{def}}{=} GF$ satisfies $E\mathscr{S} = 0$ and $Eb \neq 0$. Thus we may assume that $Fb = \alpha Fa$ for some $\alpha \in \mathbb{C}$. Put $c = b - \alpha a$ and note that $Fc = 0$. Then $c \notin \mathscr{T}$ (since $b \notin \mathscr{S}$). Hence by the inductive hypothesis there exists $H \in \mathscr{E}$ such that $H\mathscr{T} = 0$ and $Hc \neq 0$. Since $\mathscr{A}$ is a prime algebra and $Fa \neq 0$, $Hc \neq 0$, there exists $d \in \mathscr{A}$ such that $F(a)dH(c) \neq 0$. Finally, let $E \in \mathscr{E}$ be defined by

$$Ex = -F(x)dH(a) + F(a)dH(x) \qquad (x \in \mathscr{A}).$$

Then $Ea = 0$, $E\mathscr{T} = 0$ (since $F\mathscr{T} = 0$ and $H\mathscr{T} = 0$), and

$$Eb = E(c + \alpha a) = Ec = -F(c)dH(a) + F(a)dH(c) = F(a)dH(c) \neq 0. \quad \square$$

In the proof of our last two results we shall also use the following observation: if f_k $(k = 1, 2, \dots)$ is an increasing sequence of projections in a factor $\mathscr{A}$ and $e \in \mathscr{A}$ is a projection such that $f_k \prec e$ for each k, then $f \preceq e$, where $f = \bigvee_{k=1}^{\infty} f_k$. If all projections f_k are finite, this follows from [12, Lemma 2.2, p. 310]. If some f_k is infinite, then the observation follows from the fact that any two infinite cyclic projections in a factor are equivalent [4, p. 414], by noting that the cardinal number of cyclic summands in f is less than or equal to the cardinal number of cyclic summands in e (since $f = f_1 + \sum_{k=1}^{\infty}(f_{k+1} - f_k)$ and $f_{k+1} - f_k \prec e$, $f_1 \prec e$).

Proposition 2.5. *Let $\mathscr{A}$ be a von Neumann factor, $\mathscr{K}$ a closed ideal in $\mathscr{A}$, and $a_1, \dots, a_n$ elements of $\mathscr{A}$ linearly independent modulo $\mathscr{K}$. Then there exist ideals $\mathscr{J}_j$ $(j = 1, \dots, n)$ in $\mathscr{A}$ such that $\mathscr{J}_j \supset \mathscr{K}$ for each j (where the* *symbol* $\supset$ *is used in the strict sense; that is,* $\mathscr{I}_j \neq \mathscr{K}$) *and*

$$\mathscr{E}\vec{a} \supseteq \bigoplus_{j=1}^{n} \mathscr{I}_j, \tag{3}$$

where $\vec{a} = (a_1, \ldots, a_n)$.

Proof. The proof is again by an induction on n. In the case $n = 1$ it suffices to prove that the ideal $\langle a_1 \rangle$ generated by a_1 contains $\mathscr{K}$. Since $a_1 \notin \mathscr{K}$, we see (using the polar decomposition of a_1 and the spectral theorem) that the spectral projection e of $|a_1|$ corresponding to a certain positive interval satisfies $e \notin \mathscr{K}$ and $e \in \langle a_1 \rangle$. Hence it suffices to prove that $\mathscr{K} \subseteq \langle e \rangle$. For this, it suffices to show that $b \in \langle e \rangle$ for each positive b in $\mathscr{K}$. For each $k = 1, 2, \ldots$, let f_k be the spectral projection of b corresponding to the interval $[1/k, \|b\|]$. Then $f \stackrel{\text{def}}{=} \bigvee_{k=1}^{\infty} f_k$ is the range projection of b; hence, $b = fb$. From $f_k \in \mathscr{K}$, $e \notin \mathscr{K}$, and the fact that any two projections in a factor are comparable [4, p. 408], it follows that $f_k \prec e$ for each k; hence, $f \preceq e$. This implies that $f \in \langle e \rangle$; therefore, we have $b = fb \in \langle e \rangle$.

Now let $n > 1$. If we can prove that there exists an $E \in \mathscr{E}$ such that

$$Ea_j = 0 \text{ for } j = 1, \ldots, n-1 \quad \text{and} \quad Ea_n \notin \mathscr{K}, \tag{4}$$

then by a previous paragraph (applied to Ea_n) we have

$$\mathscr{E}\vec{a} \supseteq \mathscr{E}(0, \ldots, 0, Ea_n) = 0 \oplus \cdots \oplus 0 \oplus \langle Ea_n \rangle \supset 0 \oplus \cdots \oplus 0 \oplus \mathscr{K},$$

and by applying the same arguments also to other components, the proposition follows. To prove (4), assume, to the contrary, that the condition $Ea_j = 0$ for $j = 1, \ldots, n-1$ implies that $Ea_n \in \mathscr{K}$. Then the map

$$\varphi\colon \mathscr{E}(a_1, \ldots, a_{n-1}) \to \mathscr{A}/\mathscr{K},$$
$$\varphi(Ea_1, \ldots, Ea_{n-1}) \stackrel{\text{def}}{=} Ea_n + \mathscr{K} \qquad (E \in \mathscr{E})$$

is a well-defined homomorphism of $\mathscr{E}$-modules. By the inductive hypothesis we have $\mathscr{E}(a_1, \ldots, a_{n-1}) \supseteq \mathscr{K}^{n-1}$. The components φ_j of the restriction of φ to $\mathscr{K}^{n-1}$ are $\mathscr{E}$-module homomorphisms from $\mathscr{K}$ to $\mathscr{A}/\mathscr{K}$; hence, $\varphi_j = 0$ for each $j = 1, \ldots, n-1$ (since $\varphi_j(\mathscr{K}) = \varphi_j(\mathscr{K}\mathscr{K}) = \mathscr{K}\varphi_j(\mathscr{K}) = 0$ in $\mathscr{A}/\mathscr{K}$). It follows that $\varphi(\mathscr{K}^{n-1}) = 0$; hence, φ induces an $\mathscr{E}$-module homomorphism $\tilde{\varphi}\colon \mathscr{E}(a_1, \ldots, a_{n-1})/\mathscr{K}^{n-1} \to \mathscr{A}/\mathscr{K}$. Identifying in the obvious way $\mathscr{E}(a_1, \ldots, a_{n-1})/\mathscr{K}^{n-1}$ with $\widetilde{\mathscr{E}}(a_1 + \mathscr{K}, \ldots, a_{n-1} + \mathscr{K})$, where $\widetilde{\mathscr{E}}$ denotes the algebra of all elementary operators on $\mathscr{A}/\mathscr{K}$, we can regard $\tilde{\varphi}$ as an $\widetilde{\mathscr{E}}$-module homomorphism from $\widetilde{\mathscr{E}}(a_1 + \mathscr{K}, \ldots, a_{n-1} + \mathscr{K})$ to $\mathscr{A}/\mathscr{K}$ such that $\tilde{\varphi}(\widetilde{E}(a_1 + \mathscr{K}), \ldots, \widetilde{E}(a_{n-1} + \mathscr{K})) = \widetilde{E}(a_n + \mathscr{K})$ $(\widetilde{E} \in \widetilde{\mathscr{E}})$. In particular, for any fixed $\widetilde{E} \in \widetilde{\mathscr{E}}$ we have $\widetilde{E}(a_n + \mathscr{K}) = 0$ if $\widetilde{E}(a_j + \mathscr{K}) = 0$ for all $j = 1, \ldots, n-1$. Hence by Lemma 2.4 $a_n + \mathscr{K}$ must be contained in the subspace of $\mathscr{A}/\mathscr{K}$ spanned by $\{a_j + \mathscr{K} : j = 1, \ldots, n-1\}$, but this is in contradiction with the assumed linear independence modulo $\mathscr{K}$ of elements a_j $(j = 1, \ldots, n)$. (Here we have used the fact that $\mathscr{A}/\mathscr{K}$ is a prime C^*-algebra, since the closed ideals in a factor are linearly ordered by inclusion [4, p. 451].) □

Proposition 2.5 can be useful in studying the question of when the range of a fixed elementary operator is contained in a given (not necessarily closed) ideal $\mathcal{J}$ of $\mathcal{A}$. In the special case $\mathcal{J} = 0$, Mathieu [7, 9] solved the problem for general prime C^*-algebras and ultraprime Banach algebras. In the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ the question was studied in [3, 1, 2, 6]; in particular, the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ of the following corollary was proved by Apostol and Fialkow [1, Theorem 3.1].

Corollary 2.6. *Let $\mathcal{J}$ be a (not necessarily closed) ideal in a factor $\mathcal{A}$, and let E be an elementary operator on $\mathcal{A}$ defined by*

$$Ex = \sum_{j=1}^{n} a_j x b_j \qquad (x \in \mathcal{A}),$$

where $\vec{a} = (a_1, \dots, a_n)$ and $\vec{b} = (b_1, \dots, b_n)$ are fixed elements in $\mathcal{A}^n$. If $E(\mathcal{A}) \subseteq \mathcal{J}$ and $a_1, \dots, a_n$ are linearly independent modulo the norm closure $\overline{\overline{\mathcal{J}}}$ of $\mathcal{J}$ then each b_j $(j = 1, \dots, n)$ must be in $\mathcal{J}$.

Proof. We shall prove that $b_1 \in \mathcal{J}$; the proof that $b_j \in \mathcal{J}$ for $j > 1$ is the same. By Proposition 2.5 there exists an elementary operator F, say

$$Fx = \sum_{i=1}^{m} u_i x v_i \qquad (x \in \mathcal{A}),$$

where u_j, $v_j \in \mathcal{A}$ are fixed, such that

$$Fa_j = 0 \text{ for } j = 2, \dots, n \quad \text{and} \quad Fa_1 \notin \overline{\overline{\mathcal{J}}}. \tag{5}$$

Since by hypothesis $E(\mathcal{A}) \subseteq \mathcal{J}$ and $\mathcal{J}$ is an ideal, we have $\sum_{i=1}^{m} u_i E(v_i x) \in \mathcal{J}$ for all $x \in \mathcal{A}$, which can be written as

$$\sum_{i=1}^{m} u_i \sum_{j=1}^{n} a_j v_i x b_j \in \mathcal{J} \qquad (x \in \mathcal{A}).$$

Reversing the order of summation and using (5) we obtain that $(Fa_1)xb_1 \in \mathcal{J}$ for all $x \in \mathcal{A}$. Denoting Fa_1 by a and b_1 by b, we now have

$$a\mathcal{A}b \subseteq \mathcal{J} \quad \text{and} \quad a \notin \overline{\overline{\mathcal{J}}}, \tag{6}$$

and we must prove that this implies that $b \in \mathcal{J}$.

Using the polar decomposition of a and b^* and the spectral theorem for $|a| = \sqrt{a^*a}$ it follows from (6) by classical arguments (see [4, §6.8]) that for some spectral projection e of $|a|$ we have

$$e\mathcal{A}|b^*| \subseteq \mathcal{J} \quad \text{and} \quad e \notin \overline{\overline{\mathcal{J}}}.$$

For each $k = 1, 2, \dots$ let f_k be the spectral projection of $|b^*|$ corresponding to $(1/k, \infty)$. Since any two projections in a factor are comparable [4, p. 408], there are now two cases: (i) $e \preceq f_k$ for some positive integer k; (ii) $f_k \prec e$ for each k; but, from $e\mathcal{A}|b^*| \subseteq \mathcal{J}$ we have $e\mathcal{A}f_k \subseteq \mathcal{J}$ for each k. Hence in the first case it follows that $e \in \mathcal{J}$ (since $e = euf_k u^* \in \mathcal{J}u^* \subseteq \mathcal{J}$, where $u \in \mathcal{A}$ is a partial isometry with initial projection contained in f_k and final projection e), which is in contradiction with $e \notin \overline{\overline{\mathcal{J}}}$. Thus, only case (ii) occurs. Since

$f_k \prec e$ for all k, the range projection $f = \bigvee_{k=1}^{\infty} f_k$ of $|b^*|$ satisfies $f \preceq e$. Denoting by v a partial isometry with initial projection f and final projection contained in e, we have $|b^*| = f|b^*| = v^*ev|b^*| \in \mathcal{J}$ (since $e\mathcal{A}|b^*| \subseteq \mathcal{J}$). Hence $b \in \mathcal{J}$ by polar decomposition. □

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