A VECTORIAL SLEPIAN TYPE INEQUALITY. APPLICATIONS

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ABSTRACT. We prove a new inequality for Gaussian processes; this inequality implies the Chevet's inequality and Gordon's inequalities. Some remarks on Gaussian proofs of Dvoretzky's theorem are given.

I. INTRODUCTION

Let $\{g_{i,k}\}$ $(1 \le i \le n,\ 1 \le k \le d)$, $\{h_k\}_1^d$, and $\{g_i\}_1^n$ denote independent sets of orthonormal Gaussian random variables. Let E and F be Banach spaces, $\{f_k\}_{k=1}^d \subset F$ and $\{x_i^*\}_{i=1}^n \subset E^*$. Let $T(\omega) = \sum_{i=1}^n \sum_{k=1}^d g_{i,k}(\omega) x_i^* \otimes f_k$ be a random operator from E to F. The Chevet inequality says [Cv]

$$(1.1)\quad \mathbb{E}\left(\max_{\|x\|_E=1} \|T_\omega x\|\right) \le \sqrt{2}\left(\varepsilon_2(x_1^*, \dots, x_n^*)\mathbb{E}\left(\left\|\sum_{k=1}^d h_k f_k\right\|\right) + \varepsilon_2(f_1, \dots, f_d)\mathbb{E}\left(\left\|\sum_{i=1}^n g_i x_i^*\right\|_{E^*}\right)\right),$$

where

$$\varepsilon_2(x_1^*, \dots, x_n^*) = \sup\left\{\left(\sum_{1\le i\le n} x_i^*(x)^2\right)^{1/2} ;\ \|x\|_E \le 1\right\}$$

and

$$\varepsilon_2(f_1, \dots, f_d) = \sup\left\{\left(\sum_{1\le k\le d} y^*(f_k)^2\right)^{1/2} ;\ \|y^*\|_{F^*} \le 1\right\}.$$

Later, Gordon proved an inequality in the opposite direction:

(1.2)

$$\inf_{\|x\|_E=1}\left\{\left(\sum_{i=1}^n x_i^*(x)^2\right)^{1/2}\right\}\mathbb{E}\left(\left\|\sum_{k=1}^d h_k f_k\right\|\right) - \varepsilon_2(f_1, \dots, f_d)\mathbb{E}\left(\left\|\sum_{i=1}^n g_i x_i^*\right\|_{E^*}\right) \le \mathbb{E}\left(\min_{\|x\|_E=1} \|T_\omega x\|\right).$$

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He also showed that the constant $\sqrt{2}$ in (1.1) can be replaced by 1 (see [G1]). Our aim is to deduce these inequalities from a general Gaussian inequality for Gaussian processes.

II. Basic inequalities

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space and X a canonical $\mathbb{R}^d$-valued Gaussian random vector (i.e., with covariance matrix equal to Id_d). We define two Gaussian processes as follows. For $n \geq 1$, let B_2^n be the closed unit ball of l_2^n and S^{n-1} its unit sphere. For $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$, let $\|x\|_2 = (\sum_{i=1}^n (x^i)^2)^{1/2}$ and let $X_1, \ldots, X_n$ be n independent copies of X, independent of X. Let $\{g_1, \ldots, g_n\}$ be a set of orthonormal Gaussian random variables independent of $\{X, X_1, \ldots, X_n\}$. Let

$$X_x = \sum_{i=1}^n x^i X_i \quad \text{and} \quad g_x = \sum_{i=1}^n x^i g_i. \tag{2.1}$$

We shall prove the following inequality.

Theorem 2.1. *Let $A \subset B_2^n$. Let $F_x\colon \mathbb{R}^d \to \mathbb{R}$ be a family of 1-Lipschitz functions indexed by $x \in A$. Then the Gaussian processes $\{X_x\}_{x\in A}$ and $\{g_x\}_{x\in A}$ satisfy*

$$\mathbb{E} \max_{x\in A} F_x(X_x) \leq \mathbb{E} \max_{x\in A} \{F_x(\|x\|_2 X) + g_x\}. \tag{2.2}$$

Corollary 2.1. *Let $A \subset B_2^n$, and let $|||\cdot|||$ be a norm on $\mathbb{R}^d$ such that $\forall x \in \mathbb{R}^d$, $|||x||| \leq \|x\|_2$. Then the processes $\{X_x\}_{x\in A}$ and $\{g_x\}_{x\in A}$ verify*

$$\begin{aligned} \min_{x\in A} \|x\|_2 |||X||| - \mathbb{E} \max_{x\in A} g_x \leq \mathbb{E} \min_{x\in A} |||X_x||| &\leq \mathbb{E} \max_{x\in A} |||X_x||| \\ &\leq \mathbb{E}|||X||| + \mathbb{E} \max_{x\in A} g_x. \end{aligned} \tag{2.3}$$

Proof. For the right-hand side inequality put $F_y(x) = |||x|||$, and for the left-hand side inequality put $F_y(x) = -|||x|||$. □

Corollary 2.2. *Let X be a canonical $\mathbb{R}^d$-valued Gaussian random vector, with X_x and g_x as defined in (2.1). Let $A \subset S^{n-1}$, F a 1-Lipschitz function on $\mathbb{R}^d$, and $\mu = \mathbb{E}F(X)$. Then the processes $\{X_x\}_{x\in A}$ and $\{g_x\}_{x\in A}$ verify*

$$\mathbb{E} \max_{x\in A} |F(X_x) - \mu| \leq \mathbb{E}|F(X) - \mu| + \mathbb{E} \max_{x\in A} g_x \leq 1 + \mathbb{E} \max_{x\in A} g_x.$$

Proof. For the first inequality, take $G(\cdot) = |F(\cdot) - \mu|$, which is a 1-Lipschitz function; for the second, we use a well-known Poincaré-type inequality, that is,

$$\mathbb{E}|f(X) - \mathbb{E}(f(X))|^2 \leq \mathbb{E}\|\nabla f(X)\|_2^2$$

for X as above and all 1-Lipschitz functions f on $\mathbb{R}^d$ [P, C]. □

Next we show how the Gordon inequalities follow from inequality (2.3). Indeed, let $u\colon \mathbb{R}^d \to F$, $u(\sum_{k=1}^d \alpha^k e_k) = \sum_{k=1}^d \alpha^k f_k$, and $v\colon E \to l_n^2$, $v(x) = (x_1^*(x), \ldots, x_n^*(x))$. We have $\|u\| = \varepsilon_2(f_1, \ldots, f_d)$ and $\|v\| = \varepsilon_2(x_1^*, \ldots, x_n^*)$. Let $X = \sum_{k=1}^d h_k e_k$, and for $1 \leq i \leq n$ let $X_i = \sum_{k=1}^d g_{ik} e_k$. Then X is an $\mathbb{R}^d$-valued canonical Gaussian vector and $X_1, \ldots, X_n$ are n independent copies of X, independent of X. Then $u(X_{v(x)}(\omega)) = T_\omega(x)$, so the rest of the proof is as in Corollary 2.1 with $A = v(S_E)$, where S_E is the unit sphere of E and $|||\alpha||| = \|u(\alpha)\|$. □

Before proving Theorem 2.1, we get a vectorial Slepian type inequality, from which we deduce Theorem 2.1 (see Theorem 2.2).

We define some notation. For $x = (x_i)$, $y = (y_i)$ in $\mathbb{R}^d$, $x \otimes y$ denotes the matrix $(x_i y_i)_{1 \le i, j \le d}$, and for $u, v \in \mathbb{R}^d$, define $x \otimes y[u, v]$ as $\langle u, x \otimes y(v) \rangle = \langle x, u \rangle \langle y, v \rangle$ and $\| \cdot \|_{\mathscr{L}(R^d)}$ as the operator norm.

Theorem 2.2. *Let $\{X_t\}$ and $\{Y_t\}$, $t \in T$, be two families of Gaussian vectors with values in $\mathbb{R}^d$, let $\{g_t\}$ be a family of Gaussian random variables independent of $\{X_t\}$ and $\{Y_t\}$, and suppose*

(i) $\mathrm{dist}(X_t) = \mathrm{dist}(Y_t)$ *for all* $t \in T$,

(ii) $\|\mathbb{E}(X_t \otimes X_s - Y_t \otimes Y_s)\|_{\mathscr{L}(\mathbb{R}^d)} \le \frac{1}{2}\mathbb{E}|g_t - g_s|^2$ *for all* s, t *in* T.

Let F_t, $t \in T$, be a family of real 1-Lipschitz functions on $\mathbb{R}^d$. Then

$$\mathbb{E} \sup_t F_t(X_t) \le \mathbb{E} \sup_t \{F_t(Y_t) + g_t\}.$$

Proof. We may clearly assume without loss of generality that the two processes $\{X_t, t \in T\}$ and $\{Y_t, t \in T\}$ are independent and, also by a standard approximation argument, that the F_t are 1-Lipschitz and twice differentiable.

It is clear that we just need to prove the inequality for finite sets $X_1, \dots, X_N$, $Y_1, \dots, Y_N$ $(N \ge 1)$. Fix $X_1, \dots, X_N$ and $Y_1, \dots, Y_N$, and prove that

$$\mathbb{E} \max_{1 \le i \le N} \{F_i(X_i)\} \le \mathbb{E} \max_{1 \le i \le N} \{F_i(Y_i) + g_i\}. \tag{2.4}$$

For $\theta \in [0, \pi/2]$ let

$$Z(\theta) = (\cos(\theta) X_1 + \sin(\theta) Y_1, \sin(\theta) g_1; \dots; \cos(\theta) X_N + \sin(\theta) Y_N, \sin(\theta) g_N)$$

where $Z(\theta)$ is an $(\mathbb{R}^{d+1})^N$-valued Gaussian vector, with

$$Z(0) = (X_1, 0; \dots; X_N, 0) \quad \text{and} \quad Z(\pi/2) = (Y_1, g_1; \dots; Y_N, g_N);$$

a vector (y, z) of $E = (\mathbb{R}^{d+1})^N$ will be denoted by

$$(y, z) = ((y_i; z_i))_{1 \le i \le N} \quad \text{where } y_i \in \mathbb{R}^d \text{ and } z_i \in \mathbb{R}.$$

We prove first the following lemma.

Lemma 2.1. *Let $F: \mathbb{R}^{(d+1)N} \to \mathbb{R}^N$, $F(y, z) = (F_1(y_1) + z_1, \dots, F_N(y_N) + z_N)$ where $F_1, \dots, F_N$, are 1-Lipschitz twice differentiable on $\mathbb{R}^d$, and $G: \mathbb{R}^N \to \mathbb{R}$ be a twice differentiable function such that $\exists k_1, k_2$, such that $|G(\cdot)| \le k_1 e^{k_2\|\cdot\|_2}$, $|\partial G(\cdot)/\partial \alpha_i| \le k_1 e^{k_2\|\cdot\|_2}$, and $|\partial^2 G(\cdot)/\partial \alpha_i \partial \alpha_j| \le k_1 e^{k_2\|\cdot\|_2}$ for all $i, j = 1, \dots, N$. Put $\varphi = G \circ F$ and*

$$h(\theta) = \mathbb{E}\varphi(Z(\theta)). \tag{2.5}$$

Suppose

$$\forall i, j, \ i \ne j, \qquad \frac{\partial^2 G}{\partial \alpha_i \partial \alpha_j} \le 0 \tag{2.6}$$

and

$$\forall j = 1, \dots, N \qquad \sum_{i=1}^N \frac{\partial^2 G}{\partial \alpha_i \partial \alpha_j} = 0. \tag{2.7}$$

Then $h(\theta)$ is increasing; therefore,

$\mathbb{E}G(F_1(X_1), \ldots, F_N(X_N)) = h(0) \le h(\pi/2) = \mathbb{E}G(F_1(Y_1)+g_1, \ldots, F_N(Y_N)+g_N)$.

Proof of Lemma 2.1. Let $\varepsilon > 0$, and let Λ be an $(\mathbb{R}^{d+1})^N$-valued canonical Gaussian vector independent of $\{Z(\theta);\ \theta \in]0, \pi/2[\}$. Let $Z_\varepsilon(\theta) = Z(\theta) + \varepsilon\Lambda$ so that $\Gamma_\varepsilon(\theta) = \Gamma(\theta) + \varepsilon^2 I_E$, where $\Gamma(\theta)$ is the covariance matrix of $Z(\theta)$ and $\Gamma_\varepsilon(\theta)$ is the covariance matrix of $Z_\varepsilon(\theta)$. Thus

$$\Gamma_\varepsilon(\theta) \to \Gamma(\theta) \text{ as } \varepsilon \to 0 \quad \text{so that} \quad h_\varepsilon(\theta) \to h(\theta) \text{ as } \varepsilon \to 0.$$

Remark that

$$\forall (u, v) \in E \qquad \langle (u, v), \Gamma_\varepsilon(\theta)(u, v)\rangle \ge \varepsilon^2 \|(u, v)\|_E^2.$$

Let $g_\varepsilon(y, z; \theta)$ be the density function of $Z_\varepsilon(\theta)$. We will list the following well-known identities (see [G2, F, G1]):

$$(2.8)\quad g_\varepsilon(y, z; \theta) = \frac{1}{(2\pi)^{(d+1)N}} \int_E \exp\left\{ i\langle (u, v); (y, z)\rangle - \frac{1}{2}\langle (u, v), \Gamma_\varepsilon(\theta)(u, v)\rangle \right\} du\, dv$$

where $du = du_1 \cdots du_N$, $du_i = du_{i,1} \cdots du_{i,d}$, and $dv = dv_1 \cdots dv_N$;

$$(2.9)\qquad h_\varepsilon(\theta) = \int_E \varphi(y, z) g_\varepsilon(y, z, \theta)\, dy\, dz \qquad [= \mathbb{E}\varphi(Z_\varepsilon(\theta))];$$

$$(2.10)\qquad h'_\varepsilon(\theta) = \int_E \varphi(y, z) \frac{\partial}{\partial\theta} g_\varepsilon(y, z, \theta)\, dy\, dz;$$

$$(2.11)\qquad \frac{\partial}{\partial\theta} g_\varepsilon(x, \theta) = \frac{1}{2} \sum_{i,j=1}^{(d+1)N} \frac{d}{d\theta}\gamma_{i,j}^\varepsilon(\theta) \frac{\partial^2}{\partial x_i \partial x_j} g_\varepsilon(x, \theta)$$

where $x = (y, z)$ and $\Gamma_\varepsilon(\theta) = (\gamma_{i,j}^\varepsilon(\theta))_{1\le i,j\le N(d+1)}$. We compute $\Gamma_\varepsilon(\theta)$. We can write $\Gamma_\varepsilon(\theta)$ as a block matrix: $\Gamma_\varepsilon(\theta) = (\Gamma_{i,j}^\varepsilon(\theta))_{1\le i\le N,\ 1\le j\le N}$ where

$$(2.12)\qquad \Gamma_{i,j}^\varepsilon(\theta) = \mathbb{E}[Z_i^\varepsilon(\theta) \otimes Z_j^\varepsilon(\theta)]$$

where

$$Z_i^\varepsilon(\theta) = (X_i(\theta) + Y_i(\theta) + \varepsilon\Lambda_i,\ g_i(\theta) + \varepsilon\Lambda'_i)$$

where

$$\Lambda = (\Lambda_i, \Lambda'_i)_{1\le i\le N}, \qquad \Lambda_i = (\Lambda_i^1, \ldots, \Lambda_i^d),$$
$$X_i(\theta) = \cos(\theta) X_i, \qquad Y_i(\theta) = \sin(\theta) Y_i, \qquad g_i(\theta) = \sin(\theta) g_i.$$

Using the fact that $\{X_1, \ldots, X_N\}$, $\{Y_1, \ldots, Y_N\}$, and $\{g_1, \ldots, g_N\}$ are independent processes, we find that

$$(2.13)\qquad \Gamma_{i,j}^\varepsilon(\theta) = \begin{bmatrix} A_{i,j}(\theta) + \varepsilon^2 \operatorname{Id}_d \delta i, j & 0 \\ 0 & B_{i,j}^\varepsilon(\theta) \end{bmatrix}$$

where $A_{i,j}(\theta)$ is a $d \times d$ matrix and $B_{i,j}^\varepsilon(\theta)$ is a scalar such that

$$(2.14)\qquad \begin{aligned} A_{ij}(\theta) &= \cos^2(\theta)\mathbb{E}(X_i \otimes X_j) + \sin^2(\theta)\mathbb{E}(Y_i \otimes Y_j), \\ B_{ij}^\varepsilon(\theta) &= \sin^2(\theta)\mathbb{E}g_i g_j + \varepsilon^2 \delta_{i,j} \end{aligned}$$

where $\delta_{i,j} = 1$ if $i = j$, and 0 if $i \neq j$. A simple computation gives

$$\begin{aligned}&\langle (u, v); \Gamma_\varepsilon(\theta)(u, v)\rangle \\ &\quad = \sum_{i=1}^{N}\sum_{j=1}^{N}[\langle u_i, A_{i,j}(\theta)u_j\rangle + \varepsilon^2\langle u_i, u_j\rangle] + \sum_{i=1}^{N}\sum_{j=1}^{N} B_{i,j}^{\varepsilon}(\theta)v_i \cdot v_j.\end{aligned}$$

Considering $\partial^2 g_\varepsilon(y, z; \theta)/\partial y_i \partial y_j$ as a $d \times d$ matrix for each i, j gives

$$\begin{aligned}\frac{\partial}{\partial\theta} g_\varepsilon(y, z; \theta) = {}& \frac{1}{2}\sum_{i,j=1}^{N} \operatorname{trace}\left(\frac{\partial^2}{\partial y_i \partial y_j} g_\varepsilon(y, z; \theta)\frac{d}{d\theta}A_{i,j}^{\varepsilon}(\theta)\right) \\ & + \frac{1}{2}\sum_{i,j=1}^{N} \frac{d}{d\theta}B_{i,j}^{\varepsilon}(\theta)\frac{\partial^2}{\partial z_i \partial z_j} g_\varepsilon(y, z; \theta);\end{aligned}$$

but

$$h_\varepsilon'(\theta) = \int \varphi(y, z)\frac{\partial}{\partial\theta} g_\varepsilon(y, z, \theta)\, dy\, dz. \tag{2.15}$$

Let $M_{i,j} = \mathbb{E}Y_i \otimes Y_j - \mathbb{E}X_i \otimes X_j$. We get

(2.16)

$$\begin{aligned}h_\varepsilon'(\theta) = \frac{\sin 2\theta}{2}\int_E & \left\{\sum_{i,j=1}^{N} \operatorname{trace}\left(\frac{\partial^2\varphi(y, z)}{\partial y_i \partial y_j} \cdot M_{i,j}\right)\right. \\ & \left. + \sum_{i,j=1}^{N} \frac{\partial^2\varphi(y, z)}{\partial z_i \partial z_j}\mathbb{E}g_i g_j\right\} g_\varepsilon(y, z, \theta)\, dy\, dz.\end{aligned}$$

Since $\operatorname{dist}(X_i) = \operatorname{dist}(Y_i)$ for all i, we get $M_{i,i} = 0$; hence, we have, for $\varphi = G \circ F$,

$$\begin{aligned}h_\varepsilon'(\theta) = \frac{\sin 2\theta}{2}\int & \left\{\sum_{i\neq j}^{N} \operatorname{tr}\left(\frac{\partial^2 G\circ F}{\partial y_i \partial y_j} \cdot M_{i,j}\right)\right. \\ & \left. + \sum_{i,j=1}^{N} \left(\frac{\partial^2 G\circ F}{\partial z_i \partial z_j}\right)\mathbb{E}g_i g_j\right\} g_\varepsilon(y, z; \theta)\, dy\, dz.\end{aligned}$$

A simple computation gives, for all $i \neq j$,

$$\frac{\partial^2 G\circ F}{\partial y_i \partial y_j} = \frac{\partial^2 G}{\partial \alpha_i \partial \alpha_j} \circ F \cdot \nabla F_i(y_i) \otimes \nabla F_j(y_j)$$

and

$$\frac{\partial^2 G\circ F}{\partial z_i \partial z_j} = \frac{\partial^2 G}{\partial \alpha_i \partial \alpha_j} \circ F \quad \text{for all } i, j.$$

Condition (2.7) gives

$$\frac{\partial^2 G}{\partial \alpha_i^2} = -\sum_{j=1, j\neq i}^{N} \frac{\partial^2 G}{\partial \alpha_i \partial \alpha_j} \quad \text{for all } i, j, \tag{2.17}$$

so

$$h'_\varepsilon(\theta) = \frac{\sin 2\theta}{2} \int \left\{ \sum_{i\neq j}^{N} \operatorname{tr}\left(\frac{\partial^2 G(F(y\,,\,z))}{\partial y_i\,\partial y_j} \cdot M_{i\,,\,j} \right) + \sum_{i\neq j}^{N} \frac{\partial^2 G(F(y\,,\,z))}{\partial z_i\,\partial z_j} \mathbb{E} g_i g_j \right.$$
$$\left. + \sum_{i=1}^{N} \frac{\partial^2 G(F(y\,,\,z))}{\partial z_i^2} \mathbb{E} g_i^2 \right\} g_\varepsilon(y\,,\,z\,;\,\theta)\,dy\,dz$$
$$= \frac{\sin 2\theta}{2} \int \left\{ \sum_{i\neq j}^{N} \left(\operatorname{tr}\left(\frac{\partial^2 G(F(y\,,\,z))}{\partial y_i\,\partial y_j} \cdot M_{i\,,\,j} \right) \right. \right.$$
$$+ \left(\mathbb{E} g_i g_j - \frac{1}{2}[\mathbb{E} g_i^2 + \mathbb{E} g_j^2] \right)$$
$$\left. \left. \times \frac{\partial^2 G(F(y\,,\,z))}{\partial \alpha_i\,\partial \alpha_j} \right) \right\} g_\varepsilon(y\,,\,z\,;\,\theta)\,dy\,dz$$
$$= \frac{\sin 2\theta}{2} \int \left\{ \sum_{i\neq j}^{N} \left(\frac{\partial^2 G(F(y\,,\,z))}{\partial \alpha_i\,\partial \alpha_j} \left(\langle M_{i\,,\,j} \cdot \nabla F_i(y_i)\,,\, \nabla F_j(y_j) \rangle \right. \right. \right.$$
$$\left. \left. \left. - \frac{1}{2} \mathbb{E}|g_i - g_j|^2 \right) \right) \right\} g_\varepsilon(y\,,\,z\,;\,\theta)\,dy\,dz.$$

Since $\|\vec{\nabla} F_i(y_i)\| \le 1$,

$$\langle M_{i\,,\,j}(\nabla F_i(y_i))\,,\, \nabla F_j(y_j) \rangle - \tfrac{1}{2}\mathbb{E}|g_i - g_j|^2 \le \|(M_{i\,,\,j})\|_{\mathscr{L}(\mathbb{R}^d)} - \tfrac{1}{2}\mathbb{E}|g_i - g_j|^2 \le 0\,,$$

so $h'_\varepsilon(\theta) \ge 0$ and $\mathbb{E}G(F(Z_\varepsilon(0))) \le \mathbb{E}G(F(Z_\varepsilon(\pi/2)))$. Finally, letting $\varepsilon \to 0$, we get the result of Lemma 2.1. □

We now finish the proof of Theorem 2.2. The map max which assigns to each $(\alpha_1\,,\,\dots\,,\,\alpha_N) \in \mathbb{R}^N$ the value $\max(\alpha_1\,,\,\dots\,,\,\alpha_N)$ is slowly increasing and verifies (2.6) and (2.7) in distribution sense [G2]. So if we regularise max by convolution with a twice differentiable function ψ_k, which is supported by a ball of radius $1/k$, we obtain a function m_k, which is 1-Lipschitz and satisfies the above three conditions. By considering the functions $h_k(\theta) = \mathbb{E} m_k \circ F(Z(\theta))$, and by letting k go to infinity, we find by Lebesgue's theorem that the function $\mathbb{E}\max \circ F(Z(\cdot))$ is increasing in $[0\,;\,\pi/2]$. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.1. We have $X_x = \sum_{i=1}^{n} x^i X_i$. Let $Y_x = \|x\|_2 X$, where x runs over a set $A \subset B_n^2$. Then $\operatorname{dist}(X_x) = \operatorname{dist}(Y_x)$. Take a finite set $\{a_1\,,\,\dots\,,\,a_N\}$ in A; a simple computation gives

$$M_{i\,,\,j} = \mathbb{E}(Y_{a_i} \otimes Y_{a_j} - X_{a_i} \otimes X_{a_j}) = (\|a_i\|_2\|a_j\|_2 - a_i \cdot a_j)\,\mathrm{Id}_d$$

where $a_i \cdot a_j$ is the scalar product. Moreover, $\mathbb{E}|g_{a_i} - g_{a_j}|^2 = \|a_i - a_j\|_2^2$, the F_i are 1-Lipschitz functions, so

$$\|M_{i\,,\,j}\|_{\mathscr{L}(\mathbb{R}^d)} - \tfrac{1}{2}\mathbb{E}|g_{a_i} - g_{a_j}|^2 = (\|a_i\|_2\|a_j\|_2 - a_i \cdot a_j) - \tfrac{1}{2}\|a_i - a_j\|_2^2$$
$$= -\tfrac{1}{2}(\|a_i\|_2 - \|a_j\|_2)^2 \le 0.$$

Hence conditions (i) and (ii) of Theorem 2.2 are satisfied, and Theorem 2.1 is proved. □

III. Final remarks

We give now a short proof of a result due to Milman.

Theorem 3.1 [M, Sc]. *Let* $\varepsilon > 0$, $f\colon \mathbb{R}^N \to \mathbb{R}$ *be a Lipschitz function with constant* L, $X = \sum_{i=1}^N g_i e_i$ *where* $\{g_i\}_{1\le i\le N}$ *is a set of orthonormal Gaussian random variables and* $\{e_i\}_{1\le i\le N}$ *is the canonical basis of* l_2^N, *and* $\mu = \mathbb{E}f(X)$. *Then there exists an operator* $T\colon l_2^n \to \mathbb{R}^N$ *with* $n = [(\varepsilon\mu/L)(\varepsilon\mu/L - 2)]$, *such that*

$$|f(Tx) - \mu| \le \varepsilon\mu \quad \textit{for all } x \in S^{n-1}.$$

Proof. Consider, as above, real-valued Gaussian operator $T_\omega = \sum_{i=1}^n \sum_{j=1}^N g_{ij} e_i^* \otimes e_j$ from l_n^2 to $\mathbb{R}^N$,

$$X_i = \sum_{j=1}^N g_{i,j} e_j \quad \text{and} \quad X_x = \sum_{i=1}^n x^i X_i$$

where $x = (x^i, \dots, x^n)$. Then $X_x(\omega) = T_\omega x$, and we have

$$\begin{aligned}\mathbb{P}(\{\omega/\exists x \in S^{n-1};\ |f(X_x) - \mu| > \varepsilon\mu\}) &= \mathbb{P}\left(\left\{\omega;\ \sup_{x\in S^{n-1}} |f(X_x) - \mu| > \varepsilon\mu\right\}\right) \\ &\le \frac{1}{\varepsilon\mu}\mathbb{E} \sup_{x\in S^{n-1}} |f(X_x) - \mu|.\end{aligned}$$

We apply Corollary 2 to get

$$\begin{aligned}&\mathbb{P}(\{\omega/\exists x \in S^{n-1};\ |f(X_x) - \mu| > \varepsilon\mu\}) \\ &\quad\le \frac{1}{\varepsilon\mu}\left\{\mathbb{E}|f(X) - \mu| + L\mathbb{E} \sup_{x\in S^{n-1}} \sum_{j=1}^n x^j g_j\right\},\end{aligned}$$

and using the Poincaré-type inequality as in Corollary 2, we find that

$$\begin{aligned}\mathbb{P}(\{\omega/\exists x \in S^{n-1};\ |f(X_x) - \mu| > \varepsilon\mu\}) &\le \frac{L}{\varepsilon\mu}\left[1 + \mathbb{E} \sup_{x\in S^{n-1}} \sum_{j=1}^n x^j g_j\right] \\ &\le \frac{L}{\varepsilon\mu}(1 + \sqrt{n}).\end{aligned}$$

We only need to choose n such that this last expression is < 1.

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