ON APPROXIMATELY CONVEX FUNCTIONS

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Abstract. The Bernstein-Doetsch theorem on midconvex functions is extended to approximately midconvex functions and to approximately Wright convex functions.

Let $X$ be a real vector space, $D$ be a convex subset of $X$, and $\varepsilon$ be a nonnegative constant. A function $f: D \to \mathbb{R}$ is said to be

$\varepsilon$-convex if $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \varepsilon$ for all $x, y \in D$ and $t \in [0, 1]$ (cf. [2]);

$\varepsilon$-Wright-convex if $f(tx + (1 - t)y) + f((1 - t)x + ty) \leq f(x) + f(y) + 2\varepsilon$ for all $x, y \in D$ and $t \in [0, 1]$;

$\varepsilon$-midconvex if $f(\frac{1}{2}(x + y)) \leq \frac{1}{2}(f(x) + f(y)) + \varepsilon$ for all $x, y \in D$.

Notice that $\varepsilon$-convexity implies $\varepsilon$-Wright-convexity, which in turn implies $\varepsilon$-midconvexity, but not the converse. The usual notions of convexity, Wright-convexity, and midpoint convexity correspond to the case $\varepsilon = 0$. A comprehensive review on this subject can be found in [1, 6, 8-10]. The Bernstein-Doetsch theorem relates local boundedness, midconvexity, and convexity (cf. [6, 10]).

In order to extend this result to approximately midconvex functions, we first specify the assumptions on the topology $\mathcal{T}$ to be imposed on $X$: the map $(t, x, y) \to tx + y$ from $\mathbb{R} \times X \times X \to X$ is continuous in each of its three variables. Here the scalar field $\mathbb{R}$ is under the usual topology. In former literature the topology $\mathcal{T}$ is called semilinear (cf. [4, 5, 7]). These assumptions are weaker than those for $X$ to be a topological vector space. The finest $\mathcal{T}$ on $X$ is formed by taking all subsets $A \subseteq X$ with the property that if $x_0 \in A$, $x \in X$, then there exists a $\delta > 0$ such that $tx + (1 - t)x_0 \in A$ for all $t \in ] - \delta, \delta[$ [11]. In earlier literature such sets $A$ are called algebraically open (cf. also [3-5, 7]).

Lemma 1. If $D$ is convex and $f: D \to \mathbb{R}$ is $\varepsilon$-midconvex, then

$$f(k2^{-n}x + (1 - k2^{-n})y) \leq k2^{-n}f(x) + (1 - k2^{-n})f(y) + (2 - 2^{-n+1})\varepsilon$$

for all $x, y \in D$, $n \in \mathbb{N} = \{1, 2, \ldots\}$, and $k \in \{0, 1, \ldots, 2^n\}$.

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Proof. We proceed by induction. For $n = 1$, the inequality is clear. Assume that (1) holds for some $n \in \mathbb{N}$. Let $x, y \in D$ and $k \in \{0, 1, \ldots, 2^n-1\}$ be arbitrarily given. By appropriately labelling $x$ and $y$ we may assume that $k \leq 2^n$. Then we get

$$f(k2^{-n}x + (1 - k2^{-n})y) = f\left(\frac{(k2^{-n}x + (1 - k2^{-n})y) + y}{2}\right)$$

$$\leq \frac{1}{2}f(k2^{-n}x + (1 - k2^{-n})y) + \frac{1}{2}f(y) + \varepsilon$$

$$\leq \frac{1}{2}[k2^{-n}f(x) + (1 - k2^{-n})f(y) + (2 - 2^{-n+1})\varepsilon] + \frac{1}{2}f(y) + \varepsilon$$

$$= k2^{-n-1}f(x) + (1 - k2^{-n-1})f(y) + (2 - 2^{-n})\varepsilon$$

as required. This proves the lemma.

Lemma 2. Let $D$ be open and convex. If $f: D \to \mathbb{R}$ is $\varepsilon$-midconvex and locally bounded from above at a point $x_0 \in D$, then it is locally bounded from below at this point.

Proof. Let $U \subset D$ be an open set containing $x_0$ on which $f(x) \leq M$. Let $V := U \cap (2x_0 - U)$. Then $V$ is an open set containing $x_0$. Let $x \in V$ be given, and let $x' = 2x_0 - x$. Then $x' \in U$, and

$$f(x_0) = f\left(\frac{x + x'}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(x') + \varepsilon.$$

Hence $f(x) \geq 2f(x_0) - f(x') - 2\varepsilon \geq 2f(x_0) - M - 2\varepsilon$, proving that $f$ is bounded from below on $V$.

Lemma 3. Let $D$ be open and convex. If $f: D \to \mathbb{R}$ is $\varepsilon$-midconvex and locally bounded from above at a point of $D$, then it is locally bounded from above at every point of $D$.

Proof. Assume that $f$ is bounded from above on an open set $U \subset D$ containing $x_0$. Let $x \in D$ be arbitrarily given. Since $D$ is open, there exist a point $z \in D$ and a number $n \in \mathbb{N}$ such that $x = 2^{-n}x_0 + (1 - 2^{-n})z$. Put $V := 2^{-n}U + (1 - 2^{-n})z$. Then $V$ is open and contains $x$. For every $v \in V$, $v = 2^{-n}u + (1 - 2^{-n})z$ for some $u \in U$. Hence, by Lemma 1, we get $f(v) \leq 2^{-n}f(u) + (1 - 2^{-n})f(z) + 2\varepsilon$. The boundedness of $f$ from above on $V$ now follows from that of $f$ on $U$. This proves the local boundedness of $f$ from above at $x$.

Lemma 4. Let $D$ be open and convex. If $f: D \to \mathbb{R}$ is $\varepsilon$-midconvex and locally bounded from below at a point of $D$, then it is locally bounded from below at every point of $D$.

Proof. Assume that $f$ is bounded from below on an open $U \subset D$ containing $x_0$, and let $x \in D$ be arbitrarily given. Since $D$ is open, there exist a point $z \in D$ and a number $n \in \mathbb{N}$ such that $x_0 = 2^{-n}x + (1 - 2^{-n})z$. Let $V := (2^nU + (1 - 2^{-n})z) \cap D$. Then $V$ is an open neighbourhood of $x$. If $v \in V$, then $u := 2^{-n}v + (1 - 2^{-n})z \in U$, and so by Lemma 1, $f(u) \leq 2^{-n}f(v) + (1 - 2^{-n})f(z) + 2\varepsilon$. The boundedness of $f$ from below on $U$ now implies that of $f$ on $V$. This proves the local boundedness of $f$ from below at $x$.

Lemma 5. Let $D$ be a convex subset of $X$. If $f: D \to \mathbb{R}$ is $\varepsilon$-midconvex and $\delta$-convex, then it is $2\varepsilon$-convex.
Proof. Let $x \neq y$ in $D$ be arbitrarily fixed. By assumption
\[ f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \delta \quad \text{for all } t \in [0, 1]. \]

First, for $t \in [0, \frac{1}{2}]$, we obtain
\[
\begin{align*}
f(tx + (1 - t)y) &= f(\frac{1}{2}[2tx + (1 - 2t)y] + \frac{1}{2}y) \\
&\leq \frac{1}{2}f(2tx + (1 - 2t)y) + \frac{1}{2}f(y) + \varepsilon \\
&\leq \frac{1}{2}[2tf(x) + (1 - 2t)f(y) + \delta] + \frac{1}{2}f(y) + \varepsilon \\
&= f(x) + (1 - t)f(y) + \delta_1,
\end{align*}
\]
where $\delta_1 = \delta/2 + \varepsilon$. By symmetry in $x$ and $y$, the above extends to all $t \in [0, 1]$, yielding the fact that $f$ is $\delta_1$-convex. Iterating this scheme, we get that $f$ is $\delta_n$-convex for $n = 2, 3, \ldots$, where
\[ \delta_n = \frac{1}{2}\delta_{n-1} + \varepsilon. \]

Since $\delta_n \to 2\varepsilon$ as $n \to \infty$, we obtain the conclusion that $f$ is $2\varepsilon$-convex.

Theorem 1. Let $D \subset X$ be open and convex. If $f:D \to \mathbb{R}$ is $\varepsilon$-midconvex and locally bounded from above at a point of $D$, then $f$ is $2\varepsilon$-convex.

Proof. By Lemmas 2 and 3, $f$ is locally bounded from both sides at every point in $D$. Let $x \neq y$ be arbitrarily given in $D$. The segment $[x, y] = \{tx + (1-t)y : t \in [0, 1]\}$ is the image of the compact interval $[0, 1]$ under the continuous map $t \to tx + (1-t)y$, and so $[x, y]$ is compact. The local boundedness of $f$ at every point in $D$ implies that $f$ is bounded on $[x, y]$, say by $M$. This implies that the restriction of $f$ to $[x, y]$ is $2M$-convex. By Lemma 5, applied to $f$ on $[x, y]$, we get $f(tx + (1-t)y) \leq tf(x) + (1 - t)f(y) + 2\varepsilon$ for all $t \in [0, 1]$. As $x, y$ are arbitrary, this proves that $f$ is $2\varepsilon$-convex on $D$.

Corollary 1. Let $D$ be an open convex subset of $\mathbb{R}^n$, and let $f:D \to \mathbb{R}$ be $\varepsilon$-midconvex. If $f$ is bounded from above on a set $A \subset D$ of positive Lebesgue measure, then it is $2\varepsilon$-convex.

Proof. Assume that $f(x) \leq M$ for all $x \in A$. Then
\[
f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) + \varepsilon \leq M + \varepsilon \quad \text{for all } x, y \in A.
\]

Since, by the theorem of Steinhau, $\frac{1}{2}(A + A)$ has nonempty interior, the local boundedness of $f$ from above follows. Theorem 1 now yields the conclusion.

Remark. The assumption that $D$ is open in Theorem 1 is not redundant. We give an example. Let $D \subset \mathbb{R}^2$ be the closed half plane $\{(x, y) \in \mathbb{R}^2 : y \geq 0\}$, and let $f:D \to \mathbb{R}$ be given by $f(x, y) = 0$ if $y > 0$, and $f(x, y) = |a(x)|$ if $y = 0$. Here $a: \mathbb{R} \to \mathbb{R}$ is a discontinuous additive map. Then $f$ is bounded locally at each point interior to $D$ and is midconvex on $D$; however, $f$ is not convex on the $x$-axis and is, therefore, not convex on $D$.

Lemma 6. Let $I \subset \mathbb{R}$ be an interval. If $f:I \to \mathbb{R}$ is $\varepsilon$-midconvex on $I$ and $2\varepsilon$-convex in the interior of $I$, then $f$ is $2\varepsilon$-convex on $I$.

Proof. We may suppose that $I$ is not degenerated, to be interesting. Let $x \neq y$ be given in $I$. Consider $z = tx + (1-t)y$ for given $t \in [0, 1]$. Let $u = (x+z)/2$
and \( v = (z+y)/2 \). Then \( u, v \) are interior to \( I \), and \( z = tu + (1-t)v \). Hence, by \( 2\varepsilon \)-convexity in the interior of \( I \), we get

\[
f(z) \leq tf(u) + (1-t)f(v) + 2\varepsilon.
\]

Since \( f(u) \leq (f(x) + f(z))/2 + \varepsilon \) and \( f(v) \leq (f(z) + f(y))/2 + \varepsilon \) by \( \varepsilon \)-midconvexity on \( I \), we obtain

\[
f(z) \leq t \left( \frac{f(x) + f(z)}{2} + \varepsilon \right) + (1-t) \left( \frac{f(z) + f(y)}{2} + \varepsilon \right) + 2\varepsilon.
\]

This simplifies to

\[
f(z) \leq tf(x) + (1-t)f(y) + 6\varepsilon.
\]

As \( t \in [0, 1] \) is arbitrary, this proves that \( f \) is \( 6\varepsilon \)-convex on \( I \). By Lemma 5, \( f \) is \( 2\varepsilon \)-convex on \( I \).

**Theorem 2.** Let \( D \subset X \) be convex, and suppose that the boundary of \( D \) contains no proper segment \( [a, b] = \{ta + (1-t)b : t \in [0, 1]\} \) where \( a \neq b \) in \( D \). If \( f:D \to \mathbb{R} \) is \( \varepsilon \)-midconvex and is locally bounded from above at a point interior to \( D \), then \( f \) is \( 2\varepsilon \)-convex.

**Proof.** By Lemma 3, applied to the restriction of \( f \) to the interior of \( D \), we get the local boundedness of \( f \) from above at every interior point of \( D \). To show that \( f \) is \( 2\varepsilon \)-convex, we need to show that for an arbitrary given proper segment \( [a, b] \subset D \), \( f \) is \( 2\varepsilon \)-convex on \( [a, b] \). Consider pulling \([a, b]\) back to \([0, 1]\) via \( g: [0, 1] \to [a, b], g(t) = ta + (1-t)b \). Also consider \( \tilde{f} := f \circ g \)

Since \([a, b]\) contains interior points of \( D \), \( \tilde{f} \) is locally bounded from above at some interior point of \([0, 1]\). Applying Theorem 1 to \( \tilde{f} \), we get that \( \tilde{f} \) is \( 2\varepsilon \)-convex in \([0, 1]\). Applying Lemma 6, we get the \( 2\varepsilon \)-convexity of \( \tilde{f} \) on \([0, 1]\). This in turn yields that \( f \) is \( 2\varepsilon \)-convex on \([a, b]\).

**Example.** Let \( D \) be a closed ball in \( \mathbb{R}^n \) (with the usual topology). Then every \( \varepsilon \)-midconvex function \( f:D \to \mathbb{R} \), locally bounded from above at a point interior to \( D \), must be \( 2\varepsilon \)-convex. This observation extends to closed balls in a strictly convex real normed linear space.

**Lemma 7.** Let \( D \subset X \) be open and convex. If \( f:D \to \mathbb{R} \) is \( \varepsilon \)-Wright-convex and locally bounded from below at a point of \( D \), then it is \( 2\varepsilon \)-convex.

**Proof.** Let \( x \neq y \) in \( D \) be arbitrarily fixed. We need to show that \( f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + 2\varepsilon \) for all \( t \in [0, 1] \). This is an observation on the one-dimensional line passing \( x \) and \( y \); we can formally pull the problem back to the real field as follows.

Consider \( E = \{ t \in \mathbb{R} : tx + (1-t)y \in D \} \) and \( g:E \to \mathbb{R} \) given by \( g(t) = f(tx + (1-t)y) \). Since \( D \) is open and convex, so is \( E \subset \mathbb{R} \). By Lemma 4, the local boundedness of \( f \) from below at one point extends to every point of \( D \), leading to the local boundedness of \( g \) from below at every point of \( E \). Since \([0, 1]\) is compact, \( g \) is bounded from below on \([0, 1]\). The \( \varepsilon \)-Wright-convexity passes onto \( g \). In particular, we have

\[
g(1-t) + g(t) \leq g(0) + g(1) + 2\varepsilon \quad \text{for all } t \in [0, 1].
\]

As \( g(t) \) is bounded from below over all \( t \in [0, 1] \), the above implies that \( g(1-t) \) is bounded from above over all \( t \in [0, 1] \). Thus \( g \) is bounded from
above on $[0, 1]$. It follows from Theorem 1 that $g$ is $2\varepsilon$-convex on $E$. Hence
\[
f(tx + (1 - t)y) = g(t) = g(t \cdot 1 + (1 - t) \cdot 0) \\
\leq tg(1) + (1 - t)g(0) + 2\varepsilon = tf(x) + (1 - t)f(y) + 2\varepsilon,
\]
as required. This proves the lemma.

**Theorem 3.** Let $D \subset X$ be convex. If $f:D \to \mathbb{R}$ is $\varepsilon$-Wright-convex and locally bounded from below at an interior point of $D$, then it is $2\varepsilon$-convex.

**Proof.** Suppose $x_0$ is an interior point of $D$ and $f$ is locally bounded from below at $x_0$. From Lemma 7, it follows that $f$ is $2\varepsilon$-convex in the interior of $D$. Let $[x, y]$ with $x \neq y$ be a given proper segment of $D$. We need to show that $f$ is $2\varepsilon$-convex on $[x, y]$. There are two possibilities. First consider the case where $[x, y]$ contains an interior point of $D$. Then, by Lemma 4, $f$ is locally bounded from below at a point of $[x, y]$. Evidently, this implies it is locally bounded from below at a point in $]x, y[ := \{tx + (1 - t)y: 0 < t < 1\}$. Applying Lemma 7 to $f$ on $]x, y[$, or to its pull back on $]0, 1[$ if necessary, we obtain that $f$ is $2\varepsilon$-convex on $]x, y[$. Further, by Lemma 6, we obtain that $f$ is $2\varepsilon$-convex on $[x, y]$.

Second, consider the case where $[x, y]$ is on the boundary of $D$. In this case consider the triangle with vertices $x_0, x$, and $y$. By $\varepsilon$-midconvexity of $f$ we get
\[
f\left(\frac{x_0 + z}{2}\right) \leq \frac{1}{2}f(x_0) + \frac{1}{2}f(z) + \varepsilon \quad \text{for all } z \in [x, y];
\]
but the segment $\{\frac{1}{2}x_0 + \frac{1}{2}z: z \in [x, y]\}$ is in the interior of $D$ and is compact; thus $f$ is bounded from below on this segment. The above inequality implies that $f$ is bounded from below on $[x, y]$. Thus, applying Lemma 7, we first get that $f$ is $2\varepsilon$-convex on $]x, y[$ and, further by Lemma 6, obtain that $f$ is $2\varepsilon$-convex on $[x, y]$. This completes the proof.

**Remarks.** The above results remain valid when openness of a convex set $D$ is replaced by its openness relative to the manifold it generates. Theorems 1 and 3 are most forceful when the topology is the topology of algebraically open sets. Theorem 1 reduces to a result obtained by Kominek [3, Theorem 2] when $\varepsilon = 0$. The ratio $2\varepsilon$ in these two theorems is the best possible, as the following example illustrates. Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ for $x \leq 0$, and $f(x) = 1$ for $x > 0$. Then $f$ is $\varepsilon$-midconvex with lowest $\varepsilon = 1/2$. It is $\varepsilon$-convex with lowest $\varepsilon = 1$. In Theorem 1 boundedness from above cannot be replaced by boundedness from below. For example, $f(x) = |a(x)|$, where $a: \mathbb{R} \to \mathbb{R}$ is a discontinuous additive map, is midconvex on $\mathbb{R}$, and is locally bounded from below. Yet $f$ is not convex.

**References**


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