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ON APPROXIMATELY CONVEX FUNCTIONS

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ABSTRACT. The Bernstein-Doetsch theorem on midconvex functions is extended to approximately midconvex functions and to approximately Wright convex functions.

Let X be a real vector space, D be a convex subset of X, and ε be a nonnegative constant. A function $f: D \to \mathbb{R}$ is said to be

ε-convex if $f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+\varepsilon$ for all $x, y \in D$ and $t \in [0, 1]$ (cf. [2]);

ε-Wright-convex if $f(tx+(1-t)y)+f((1-t)x+ty) \le f(x)+f(y)+2\varepsilon$ for all $x, y \in D$ and $t \in [0, 1]$;

ε-midconvex if $f(\frac{x+y}{2}) \le \frac{1}{2}(f(x)+f(y))+\varepsilon$ for all $x, y \in D$.

Notice that ε-convexity implies ε-Wright-convexity, which in turn implies ε-midconvexity, but not the converse. The usual notions of convexity, Wright-convexity, and midconvexity correspond to the case $\varepsilon = 0$. A comprehensive review on this subject can be found in [1, 6, 8–10]. The Bernstein-Doetsch theorem relates local boundedness, midconvexity, and convexity (cf. [6, 10]). In order to extend this result to approximately midconvex functions, we first specify the assumptions on the topology $\mathscr{T}$ to be imposed on X: the map $(t, x, y) \to tx+y$ from $\mathbb{R} \times X \times X \to X$ is continuous in each of its three variables. Here the scalar field $\mathbb{R}$ is under the usual topology. In former literature the topology $\mathscr{T}$ is called *semilinear* (cf. [4, 5, 7]). These assumptions are weaker than those for X to be a topological vector space. The finest $\mathscr{T}$ on X is formed by taking all subsets $A \subset X$ with the property that if $x_0 \in A$, $x \in X$, then there exists a $\delta > 0$ such that $tx+(1-t)x_0 \in A$ for all $t \in]-\delta, \delta[$. In earlier literature such sets A are called *algebraically open* [11] (cf. also [3–5, 7]).

Lemma 1. *If D is convex and $f: D \to \mathbb{R}$ is ε-midconvex, then*

$$(1) \qquad f(k2^{-n}x+(1-k2^{-n})y) \le k2^{-n}f(x)+(1-k2^{-n})f(y)+(2-2^{-n+1})\varepsilon$$

for all $x, y \in D$, $n \in \mathbb{N} = \{1, 2, \dots\}$, and $k \in \{0, 1, \dots, 2^n\}$.

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Proof. We proceed by induction. For $n = 1$, the inequality is clear. Assume that (1) holds for some $n \in \mathbb{N}$. Let $x, y \in D$ and $k \in \{0, 1, \dots, 2^{n+1}\}$ be arbitrarily given. By appropriately labelling x and y we may assume that $k \le 2^n$. Then we get

$$\begin{aligned} f(k2^{-n-1}x + (1 - k2^{-n-1})y) &= f\left(\frac{(k2^{-n}x + (1 - k2^{-n})y) + y}{2}\right) \\ &\le \frac{1}{2}f(k2^{-n}x + (1 - k2^{-n})y) + \frac{1}{2}f(y) + \varepsilon \\ &\le \frac{1}{2}[k2^{-n}f(x) + (1 - k2^{-n})f(y) + (2 - 2^{-n+1})\varepsilon] + \frac{1}{2}f(y) + \varepsilon \\ &= k2^{-n-1}f(x) + (1 - k2^{-n-1})f(y) + (2 - 2^{-n})\varepsilon \end{aligned}$$

as required. This proves the lemma.

Lemma 2. *Let D be open and convex. If $f: D \to \mathbb{R}$ is ε-midconvex and locally bounded from above at a point $x_0 \in D$, then it is locally bounded from below at this point.*

Proof. Let $U \subset D$ be an open set containing x_0 on which $f(x) \le M$. Let $V := U \cap (2x_0 - U)$. Then V is an open set containing x_0. Let $x \in V$ be given, and let $x' = 2x_0 - x$. Then $x' \in U$, and

$$f(x_0) = f\left(\frac{x + x'}{2}\right) \le \frac{1}{2}f(x) + \frac{1}{2}f(x') + \varepsilon.$$

Hence $f(x) \ge 2f(x_0) - f(x') - 2\varepsilon \ge 2f(x_0) - M - 2\varepsilon$, proving that f is bounded from below on V.

Lemma 3. *Let D be open and convex. If $f: D \to \mathbb{R}$ is ε-midconvex and locally bounded from above at a point of D, then it is locally bounded from above at every point of D.*

Proof. Assume that f is bounded from above on an open set $U \subset D$ containing x_0. Let $x \in D$ be arbitrarily given. Since D is open, there exist a point $z \in D$ and a number $n \in \mathbb{N}$ such that $x = 2^{-n}x_0 + (1 - 2^{-n})z$. Put $V := 2^{-n}U + (1 - 2^{-n})z$. Then V is open and contains x. For every $v \in V$, $v = 2^{-n}u + (1 - 2^{-n})z$ for some $u \in U$. Hence, by Lemma 1, we get $f(v) \le 2^{-n}f(u) + (1 - 2^{-n})f(z) + 2\varepsilon$. The boundedness of f from above on V now follows from that of f on U. This proves the local boundedness of f from above at x.

Lemma 4. *Let D be open and convex. If $f: D \to \mathbb{R}$ is ε-midconvex and locally bounded from below at a point of D, then it is locally bounded from below at every point of D.*

Proof. Assume that f is bounded from below on an open $U \subset D$ containing x_0, and let $x \in D$ be arbitrarily given. Since D is open, there exist a point $z \in D$ and a number $n \in \mathbb{N}$ such that $x_0 = 2^{-n}x + (1 - 2^{-n})z$. Let $V := (2^nU + (1 - 2^n)z) \cap D$. Then V is an open neighbourhood of x. If $v \in V$, then $u := 2^{-n}v + (1 - 2^{-n})z \in U$, and so by Lemma 1, $f(u) \le 2^{-n}f(v) + (1 - 2^{-n})f(z) + 2\varepsilon$. The boundedness of f from below on U now implies that of f on V. This proves the local boundedness of f from below at x.

Lemma 5. *Let D be a convex subset of X. If $f: D \to \mathbb{R}$ is ε-midconvex and δ-convex, then it is 2ε-convex.*

Proof. Let $x \neq y$ in D be arbitrarily fixed. By assumption

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \delta \quad \text{for all } t \in [0, 1].$$

First, for $t \in [0, \frac{1}{2}]$, we obtain

$$\begin{aligned} f(tx + (1-t)y) &= f(\tfrac{1}{2}[2tx + (1-2t)y] + \tfrac{1}{2}y) \\ &\leq \tfrac{1}{2}f(2tx + (1-2t)y) + \tfrac{1}{2}f(y) + \varepsilon \\ &\leq \tfrac{1}{2}[2tf(x) + (1-2t)f(y) + \delta] + \tfrac{1}{2}f(y) + \varepsilon \\ &= tf(x) + (1-t)f(y) + \delta_1, \end{aligned}$$

where $\delta_1 = \delta/2 + \varepsilon$. By symmetry in x and y, the above extends to all $t \in [0, 1]$, yielding the fact that f is δ_1-convex. Iterating this scheme, we get that f is δ_n-convex for $n = 2, 3, \ldots$, where

$$\delta_n = \tfrac{1}{2}\delta_{n-1} + \varepsilon.$$

Since $\delta_n \to 2\varepsilon$ as $n \to \infty$, we obtain the conclusion that f is 2ε-convex.

Theorem 1. *Let $D \subset X$ be open and convex. If $f: D \to \mathbb{R}$ is ε-midconvex and locally bounded from above at a point of D, then f is 2ε-convex.*

Proof. By Lemmas 2 and 3, f is locally bounded from both sides at every point in D. Let $x \neq y$ be arbitrarily given in D. The segment $[x, y] = \{tx + (1-t)y : t \in [0, 1]\}$ is the image of the compact interval $[0, 1]$ under the continuous map $t \to tx + (1-t)y$, and so $[x, y]$ is compact. The local boundedness of f at every point in D implies that f is bounded on $[x, y]$, say by M. This implies that the restriction of f to $[x, y]$ is $2M$-convex. By Lemma 5, applied to f on $[x, y]$, we get $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + 2\varepsilon$ for all $t \in [0, 1]$. As x, y are arbitrary, this proves that f is 2ε-convex on D.

Corollary 1. *Let D be an open convex subset of $\mathbb{R}^n$, and let $f: D \to \mathbb{R}$ be ε-midconvex. If f is bounded from above on a set $A \subset D$ of positive Lebesgue measure, then it is 2ε-convex.*

Proof. Assume that $f(x) \leq M$ for all $x \in A$. Then

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) + \varepsilon \leq M + \varepsilon \quad \text{for all } x, y \in A.$$

Since, by the theorem of Steinhaus, $\frac{1}{2}(A + A)$ has nonempty interior, the local boundedness of f from above follows. Theorem 1 now yields the conclusion.

Remark. The assumption that D is open in Theorem 1 is not redundant. We give an example. Let $D \subset \mathbb{R}^2$ be the closed half plane $\{(x, y) \in \mathbb{R}^2 : y \geq 0\}$, and let $f: D \to \mathbb{R}$ be given by $f(x, y) = 0$ if $y > 0$, and $f(x, y) = |a(x)|$ if $y = 0$. Here $a: \mathbb{R} \to \mathbb{R}$ is a discontinuous additive map. Then f is bounded locally at each point interior to D and is midconvex on D; however, f is not convex on the x-axis and is, therefore, not convex on D.

Lemma 6. *Let $I \subset \mathbb{R}$ be an interval. If $f: I \to \mathbb{R}$ is ε-midconvex on I and 2ε-convex in the interior of I, then f is 2ε-convex on I.*

Proof. We may suppose that I is not degenerated, to be interesting. Let $x \neq y$ be given in I. Consider $z = tx + (1-t)y$ for given $t \in]0, 1[$. Let $u = (x+z)/2$

and $v = (z+y)/2$. Then u, v are interior to I, and $z = tu+(1-t)v$. Hence, by 2ε-convexity in the interior of I, we get

$$f(z) \leq tf(u) + (1-t)f(v) + 2\varepsilon.$$

Since $f(u) \leq [f(x) + f(z)]/2 + \varepsilon$ and $f(v) \leq [f(z) + f(y)]/2 + \varepsilon$ by ε-midconvexity on I, we obtain

$$f(z) \leq t\left[\frac{f(x)+f(z)}{2} + \varepsilon\right] + (1-t)\left[\frac{f(z)+f(y)}{2} + \varepsilon\right] + 2\varepsilon.$$

This simplifies to

$$f(z) \leq tf(x) + (1-t)f(y) + 6\varepsilon.$$

As $t \in \,]0, 1[$ is arbitrary, this proves that f is 6ε-convex on I. By Lemma 5, f is 2ε-convex on I.

Theorem 2. *Let $D \subset X$ be convex, and suppose that the boundary of D contains no proper segment $[a, b] = \{ta + (1-t)b : t \in [0, 1]\}$ where $a \neq b$ in D. If $f: D \to \mathbb{R}$ is ε-midconvex and is locally bounded from above at a point interior to D, then f is 2ε-convex.*

Proof. By Lemma 3, applied to the restriction of f to the interior of D, we get the local boundedness of f from above at every interior point of D. To show that f is 2ε-convex, we need to show that for an arbitrary given proper segment $[a, b] \subset D$, f is 2ε-convex on $[a, b]$. Consider pulling $[a, b]$ back to $[0, 1]$ via $g: [0, 1] \to [a, b]$, $g(t) = ta+(1-t)b$. Also consider $\overline{f} := f \circ g$. Since $[a, b]$ contains interior points of D, $\overline{f}$ is locally bounded from above at some interior point of $[0, 1]$. Applying Theorem 1 to $\overline{f}$, we get that $\overline{f}$ is 2ε-convex in $]0, 1[$. Applying Lemma 6, we get the 2ε-convexity of $\overline{f}$ on $[0, 1]$. This in turn yields that f is 2ε-convex on $[a, b]$.

Example. Let D be a closed ball in $\mathbb{R}^n$ (with the usual topology). Then every ε-midconvex function $f: D \to \mathbb{R}$, locally bounded from above at a point interior to D, must be 2ε-convex. This observation extends to closed balls in a strictly convex real normed linear space.

Lemma 7. *Let $D \subset X$ be open and convex. If $f: D \to \mathbb{R}$ is ε-Wright-convex and locally bounded from below at a point of D, then it is 2ε-convex.*

Proof. Let $x \neq y$ in D be arbitrarily fixed. We need to show that $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + 2\varepsilon$ for all $t \in [0, 1]$. This is an observation on the one-dimensional line passing x and y; we can formally pull the problem back to the real field as follows.

Consider $E = \{t \in \mathbb{R}: tx + (1-t)y \in D\}$ and $g: E \to \mathbb{R}$ given by $g(t) = f(tx + (1-t)y)$. Since D is open and convex, so is $E \subset \mathbb{R}$. By Lemma 4, the local boundedness of f from below at one point extends to every point of D, leading to the local boundedness of g from below at every point of E. Since $[0, 1]$ is compact, g is bounded from below on $[0, 1]$. The ε-Wright-convexity passes onto g. In particular, we have

$$g(1-t) + g(t) \leq g(0) + g(1) + 2\varepsilon \quad \text{for all } t \in [0, 1].$$

As $g(t)$ is bounded from below over all $t \in [0, 1]$, the above implies that $g(1-t)$ is bounded from above over all $t \in [0, 1]$. Thus g is bounded from above on $[0, 1]$. It follows from Theorem 1 that g is 2ε-convex on E. Hence

$$\begin{aligned} f(tx+(1-t)y) &= g(t) = g(t\cdot 1 + (1-t)\cdot 0) \\ &\le tg(1) + (1-t)g(0) + 2\varepsilon = tf(x) + (1-t)f(y) + 2\varepsilon, \end{aligned}$$

as required. This proves the lemma.

Theorem 3. *Let $D \subset X$ be convex. If $f: D \to \mathbb{R}$ is ε-Wright-convex and locally bounded from below at an interior point of D, then it is 2ε-convex.*

Proof. Suppose x_0 is an interior point of D and f is locally bounded from below at x_0. From Lemma 7, it follows that f is 2ε-convex in the interior of D. Let $[x, y]$ with $x \neq y$ be a given proper segment of D. We need to show that f is 2ε-convex on $[x, y]$. There are two possibilities. First consider the case where $[x, y]$ contains an interior point of D. Then, by Lemma 4, f is locally bounded from below at a point of $[x, y]$. Evidently, this implies it is locally bounded from below at a point in $]x, y[:= \{tx + (1-t)y : 0 < t < 1\}$. Applying Lemma 7 to f on $]x, y[$, or to its pull back on $]0, 1[$ if necessary, we obtain that f is 2ε-convex on $]x, y[$. Further, by Lemma 6, we obtain that f is 2ε-convex on $[x, y]$.

Second, consider the case where $[x, y]$ is on the boundary of D. In this case consider the triangle with vertices x_0, x, and y. By ε-midconvexity of f we get

$$f\left(\frac{x_0+z}{2}\right) \le \frac{1}{2}f(x_0) + \frac{1}{2}f(z) + \varepsilon \quad \text{for all } z \in [x, y];$$

but the segment $\{\frac{1}{2}x_0 + \frac{1}{2}z : z \in [x, y]\}$ is in the interior of D and is compact; thus f is bounded from below on this segment. The above inequality implies that f is bounded from below on $[x, y]$. Thus, applying Lemma 7, we first get that f is 2ε-convex on $]x, y[$ and, further by Lemma 6, obtain that f is 2ε-convex on $[x, y]$. This completes the proof.

Remarks. The above results remain valid when openness of a convex set D is replaced by its openness relative to the manifold it generates. Theorems 1 and 3 are most forceful when the topology is the topology of algebraically open sets. Theorem 1 reduces to a result obtained by Kominek [3, Theorem 2] when $\varepsilon = 0$. The ratio 2ε in these two theorems is the best possible, as the following example illustrates. Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ for $x \le 0$, and $f(x) = 1$ for $x > 0$. Then f is ε-midconvex with lowest $\varepsilon = 1/2$. It is ε-convex with lowest $\varepsilon = 1$. In Theorem 1 boundedness from above cannot be replaced by boundedness from below. For example, $f(x) = |a(x)|$, where $a: \mathbb{R} \to \mathbb{R}$ is a discontinuous additive map, is midconvex on $\mathbb{R}$, and is locally bounded from below. Yet f is not convex.

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