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ON MAPPINGS WITH INTEGRABLE DILATATION

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ABSTRACT. A factorization of Stoilow's type has been obtained for mappings in $\mathbb{R}^2$ with integrable dilatation.

0. INTRODUCTION

For Ω a domain in $\mathbb{R}^n$ (an open and connected set), we consider a mapping $f: \Omega \to \mathbb{R}^n$ of the Sobolev class $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with nonnegative Jacobian, $J(x, f) \geq 0$ a.e. We say that f has finite dilatation if

$$|Df(x)|^n \leq K(x)J(x, f) \quad \text{a.e.} \tag{0.1}$$

where $1 \leq K(x) < \infty$ for almost every $x \in \Omega$ and $|Df(x)|$ denotes the norm of the differential $Df(x): \mathbb{R}^n \to \mathbb{R}^n$.

In recent developments of the nonlinear elasticity theory [Ba, Š], there have been intensive studies of the analytic and geometric properties of such mappings. It is known that the condition $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ does not guarantee that f is continuous, but it does if f has finite dilatation [VG], see also [Ma] for a simpler proof. To state our result we need some definitions.

The dilatation quotient at the points $x \in \Omega$ with $J(x, f) \neq 0$ is defined by

$$K(x, f) = \frac{|Df(x)|^n}{J(x, f)} \geq 1. \tag{0.2}$$

If $J(x, f) = 0$, then $Df(x) = 0$, and in this case we put $K(x, f) = 1$ a.e. Therefore the dilatation function $K(\ , f): \Omega \to [1, \infty)$ is defined almost everywhere in Ω. A mapping $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is said to be K-quasi-regular, $1 \leq K < \infty$, if $K(x, f) \leq K$ a.e. If, in addition, f is a homeomorphism, we say that f is K-quasi-conformal.

A well-known result in the theory of quasi-regular mappings [Re] states that if $K(\ , f) \in L^\infty(\Omega)$, then f is either constant or an open mapping. In two

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dimensions this fact has already been recognized by Bojarski [Bo1, Bo2], who has proved Stoilow's type factorization

$$f = \varphi \circ h^{-1}, \tag{0.3}$$

with $h: \Omega' \to \Omega$ a homeomorphism (quasi-conformal mapping) and $\varphi: \Omega' \to \mathbb{R}^2$ a holomorphic function.

In this note we prove that this factorization remains valid for 2-dimensional mappings whose dilatation function is only assumed to be integrable. Such mappings are, therefore, open and discrete.

Theorem 1. *Let Ω be a bounded domain in the complex plane $(\mathbb{C}, d\sigma(z))$ and, $f \in W^{1,2}(\Omega, \mathbb{C})$ with $J(z, f) \geq 0$ and $K(\ , f) \in L^1(\Omega)$. Then there exists a homeomorphism $h \in W^{1,2}(\Omega', \Omega)$ and a holomorphic function $\varphi \in W^{1,2}(\Omega', \mathbb{C})$ such that*

$$f = \varphi \circ h^{-1}. \tag{0.4}$$

Moreover,

$$\int_{\Omega'} |Dh(\omega)|^2 \, d\sigma(\omega) \leq \int_{\Omega} K(z, f) \, d\sigma(z) \tag{0.5}$$

and

$$\int_{\Omega'} |\varphi'(\omega)|^2 \, d\sigma(\omega) \leq \int_{\Omega} \left| \frac{\partial}{\partial z} f(z) \right|^2 d\sigma(z). \tag{0.6}$$

One can ask whether some integrability condition on the dilatation function of a mapping $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ with positive Jacobian implies openness also in dimension $n > 2$. The arguments we have used in the proof of Theorem 1 suggest the following.

Conjecture 1. *Let Ω be a domain in $\mathbb{R}^n$ and $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ with $J(x, f) \geq 0$ and $K(\ , f) \in L^{n-1}(\Omega)$. Then f is either constant or an open mapping.*

This has already been shown under additional assumptions about the boundary values of f [Ba, Š]. The general case still remains open.[1]

1. Preliminaries

We need a few of the fundamental properties of the quasi-regular mapping in $\mathbb{R}^n$. Let us recall the chain rule for differentiation of the composite functions [BI].

Lemma 1.1. *Let $f \in W^{1,n}_{\text{loc}}(\Omega, \Omega')$ be a quasi-regular mapping and let $\varphi \in W^{1,n}_{\text{loc}}(\Omega')$. Then $\varphi \circ f \in W^{1,n}_{\text{loc}}(\Omega)$, and*

$$D(\varphi \circ f)(x) = (D\varphi)(f(x)) \circ Df(x) \tag{1.1}$$

for almost every $x \in \Omega$.

The next result concerns the change of variables in a multiple integral [BI, Re, Ri].

[1] Very recently J. Heinonen and P. Koskela confirmed this conjecture for f a "quasi-light mapping" with $K(\ , f) \in L^p_{\text{loc}}(\Omega)$ and $p > n-1$, and most recently for $f \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n)$ with $p > n$ and $K \in L^{p/(n-1)(p-n)}_{\text{loc}}(\Omega)$.

Lemma 1.2. *Let* $f\colon \Omega \to \Omega'$ *be quasi-conformal and* $u \in L^1(\Omega')$. *Then* $u(f(x))J(x, f) \in L^1(\Omega)$, *and we have*

$$\int_{\Omega'} u(y)\, dy = \int_{\Omega} u(f(x))J(x, f)\, dx. \tag{1.2}$$

With the aid of these two lemmas we easily arrive at an estimate of the L^n-norm of the differential of the inverse mapping $h = f^{-1}\colon \Omega' \to \Omega$ in terms of the dilatation function of f.

Lemma 1.3. *Let* $f\colon \Omega \to \Omega'$ *be a quasi-conformal mapping of bounded domains* Ω, $\Omega' \subset \mathbb{R}^n$, *and let* $h\colon \Omega' \to \Omega$ *denote the inverse mapping. Then*

$$\int_{\Omega'} |Dh(y)|^n\, dy \le \int_{\Omega} K^{n-1}(x, f)\, dx. \tag{1.3}$$

Proof. We have

$$\begin{aligned}\int_{\Omega'} |Dh|^n &= \int_{\Omega} |D(f)^{-1}(x)|^n J(x, f)\, dx = \int_{\Omega} |\operatorname{adj} Df(x)|^n J^{1-n}(x, f)\, dx \\ &\le \int_{\Omega} |Df(x)|^{n(n-1)} J^{1-n}(x, f)\, dx = \int_{\Omega} K^{n-1}(x, f)\, dx,\end{aligned}$$

as desired.

This is why we assumed in Conjecture 1 that $K(\ , f) \in L^{n-1}(\Omega)$. The last prerequisite deals with the concept and properties of monotone mappings.

We refer to the article of McAuley [McA], in which this subject is well covered by a series of papers.

Let X and Y be compact metric spaces. A continuous mapping h from X onto Y is said to be monotone if for each $y \in Y$ the set $f^{-1}(y)$ is connected. Actually, as shown by Whyburn, this implies that $f^{-1}(C)$ is connected for each connected set C in Y.

We shall use the following result of Kuratowski, Lacher, and Whyburn [McA].

Lemma 1.4. *If* Y *is locally connected, then the set of all monotone mappings from* X *onto* Y *is closed in* $C(X, Y)$. *The latter stands for the space of all continuous mappings of* X *into* Y *with the topology of uniform convergence.*

In our application of this result X and Y will be the 2-spheres, in which case Lemma 1.4 is an elementary exercise.

2. The Beltrami equation

The space $\mathbb{R}^2$ will be identified with the complex plane $\mathbb{C}$, where the area element is denoted by $d\sigma(z) = dx\, dy$, $z = x + iy$. For $a \in \mathbb{C}$ and $r > 0$, we define the open disk $B(a, r) = \{z\,;\, |z-a| < r\}$ and its boundary $S(a, r) = \{z\,;\, |z-a| = r\}$.

On the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ we introduce the chordal distance

$$d(a, \infty) = \frac{2}{\sqrt{1+|a|^2}} \quad \text{and} \quad d(a, b) = \frac{2|a-b|}{\sqrt{1+|a|^2}\sqrt{1+|b|^2}}$$

if a, $b \neq \infty$.

Thus $\widehat{\mathbb{C}}$ is a compact metric space. Recall that the chordal distance d is inherited from the Euclidean metric on the 2-sphere via the stereographic projection. It is clear that d restricted to $\mathbb{C}$ induces the same topology as does the Euclidean metric.

We shall make use of the Cauchy-Riemann derivatives. For $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{C})$, where Ω is an open subset of $\mathbb{C}$, the derivatives are defined by

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad f_{\bar z} = \frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

It is straightforward to check the formulas

$$J(z, f) = |f_z|^2 - |f_{\bar z}|^2 \quad \text{and} \quad |Df(z)| = |f_z| + |f_{\bar z}|.$$

Our proof of Theorem 1 will rest on the existence theorem for the Beltrami equation.

Proposition 2.1 (Bojarski [Bo1, Bo2]). *Let μ be an arbitrary measurable function with compact support and $\|\mu\|_\infty < 1$. Then, for some $p > 2$, there exists a unique solution $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{C}, \mathbb{C})$ of the Beltrami equation*

$$f_{\bar z}(z) = \mu(z) f_z(z) \tag{2.1}$$

such that $f(0) = 0$ and $1 - f_z \in L^p(\mathbb{C})$.

This is what we call the normal solution of (2.1) The coefficient μ is referred to as the complex dilatation of f. The normal solution is a quasi-conformal homeomorphism of the extended complex plane, analytic outside the support of μ, and its Taylor expansion at infinity takes the form $f(z) = z + a_1 z^{-1} + a_2 z^{-2} = \cdots$. See also [A, L, LV].

For results concerning the existence of solutions of (2.1) with $\|\mu\|_\infty = 1$, we refer to David [D]; see also [P].

From now on we confine ourselves to those μ which are supported in the unit disk $\mathbb{B} = \{z;\ |z| < 1\}$. The purpose of this section is to establish uniform estimates for the inverse mapping $h = f^{-1}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. First, recall the inequality

$$\int_E |Dh(\omega)|^2 \, d\sigma(\omega) \le \int_{h(E)} K(z, f)\, d\sigma(z) \tag{2.2}$$

for each measurable set $E \subset \mathbb{C}$. Here, the dilatation function of f can be expressed in terms of μ as

$$K(z, f) = \frac{|Df(z)|^2}{J(z, f)} = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}.$$

Proposition 2.2. *Let $B_r = B(0, r)$, $r > 1$, and ξ, $\zeta \in B_r$ be such that $|\xi - \zeta| < 2$. Then*

$$|h(\xi) - h(\zeta)|^2 \log \frac{4}{|\xi - \zeta|^2} \le \pi \int_{B_{r+3}} K(z, f)\, d\sigma(z). \tag{2.3}$$

The chordal distance from $h(\xi)$ to ∞ is estimated independently of μ as

$$d(h(\xi), \infty) \le \frac{10}{1 + |\xi|} \tag{2.4}$$

for all $\xi \in \widehat{\mathbb{C}}$.

Proof. Take notice that f maps $\widehat{\mathbb{C}}-\mathbb{B}$ univalently into $\widehat{\mathbb{C}}-\{0\}$. In view of the Koebe distortion inequality, we can write $|z|+|z|^{-1}-2 \le |f(z)| \le |z|+|z|^{-1}+2$, for all z with $|z|>1$; see, for example, [M]. Hence, $B_{r+1} \subset f(B_{r+3})$, which is equivalent to

$$h(B_{r+1}) \subset B_{r+3}. \tag{2.5}$$

We also infer that

$$|f(z)| \le 4|z| \quad \text{for } |z| \ge 1. \tag{2.6}$$

To prove inequality (2.3) we set $|\xi-\zeta| = 2\delta < 2$ and $a = \frac{1}{2}(\xi+\zeta) \in B_r$. Obviously, $S(a, \delta) \subset B(a, t) \subset B_{r+1}$ if $\delta < t < 1$. Since h is a homeomorphism, by the maximum principle, one can find points ξ', $\zeta' \in S(a, t)$ such that $|h(\xi)-h(\zeta)| \le |h(\xi')-h(\zeta')|$. The latter is easily estimated by the integral of $|Dh|$ over the circle $S(a, t)$:

$$|h(\xi)-h(\zeta)| \le \frac{1}{2}\int_{S(a,t)} |Dh(\omega)|\,|d\omega| \tag{2.7}$$

for almost every $t \in (\delta, 1)$. By Hölder's inequality we obtain

$$t^{-1}|h(\xi)-h(\zeta)|^2 \le \frac{\pi}{2}\int_{S(a,t)} |Dh(\omega)|^2|d\omega|.$$

Integrating with respect to $t \in (\delta, 1)$, by Fubini's theorem, we find that

$$\begin{aligned} -2|h(\xi)-h(\zeta)|^2\log\delta &\le \pi\int_{B(a,1)} |Dh(\omega)|^2\,d\sigma(\omega) \\ &\le \pi\int_{B_{r+1}} |Dh(\omega)|^2\,d\sigma(\omega). \end{aligned}$$

This, together with (2.2) and (2.5), yields

$$\begin{aligned} |h(\xi)-h(\zeta)|^2\log\frac{4}{|\xi-\zeta|^2} &\le \pi\int_{h(B_{r+1})} K(z, f)\,d\sigma(z) \\ &\le \pi\int_{B_{r+3}} K(z, f)\,d\sigma(z), \end{aligned}$$

as desired.

Concerning estimate (2.4), it is equivalent to show that $d(z, \infty) \le 10/(1+|f(z)|)$ for all $z \in \widehat{\mathbb{C}}$. If $|z|>1$ we use (2.6) to obtain

$$d(z, \infty) = \frac{2}{\sqrt{1+|z|^2}} \le \frac{5\sqrt{2}}{1+4|z|} \le \frac{10}{1+|f(z)|}.$$

For $|z|<1$, in view of the maximum principle $|f(z)| \le \max_{|\xi|=1}|f(\xi)| \le 4$, we conclude

$$\rho(z, \infty) \le 2 \le \frac{10}{1+|f(z)|}.$$

This completes the proof of Proposition 2.2.

3. Proof of Theorem 1

We may assume that Ω is a subdomain of the unit disk, $\Omega \subset \mathbb{B}$, and $f \neq$ constant. Consider the complex dilatation $\mu = \mu(z)$ of f; that is,

$$f_{\bar z} = \mu(z) f_z \quad \text{a.e. in } \Omega. \tag{3.1}$$

We extend μ by zero outside Ω and regard it as a function on the whole of $\mathbb{C}$. For $0 < \varepsilon < 1$ we define

$$\mu^\varepsilon(z) = \begin{cases} \mu(z) & \text{if } |\mu(z)| \le 1 - \varepsilon, \\ (1-\varepsilon)\mu(z)|\mu(z)|^{-1} & \text{if } |\mu(z)| > 1 - \varepsilon. \end{cases} \tag{3.2}$$

Let $f^\varepsilon\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the normal solution of the Beltrami equation

$$f^\varepsilon_{\bar z} = \mu^\varepsilon(z) f^\varepsilon_z, \tag{3.3}$$

and let $h^\varepsilon = (f^\varepsilon)^{-1}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ denote the inverse mapping. With the aid of Lemma 1.1 we find that

$$f^\varepsilon_{\bar z} = -J(z, f^\varepsilon) h^\varepsilon_{\bar\omega} \quad\text{and}\quad f^\varepsilon_z = J(z, f^\varepsilon)\overline{h^\varepsilon_\omega}.$$

Therefore, (3.3) becomes a quasi-linear equation for h^ε (the hodograph transformation)

$$h^\varepsilon_{\bar\omega} = -\mu^\varepsilon(h^\varepsilon(\omega))\overline{h^\varepsilon_\omega}. \tag{3.4}$$

The dilatation function of f^ε can be estimated independently of ε as

$$K(z, f^\varepsilon) = \frac{1 + |\mu^\varepsilon(z)|}{1 - |\mu^\varepsilon(z)|} \le \frac{1 + |\mu(z)|}{1 - |\mu(z)|} = K(z, f),$$

where, in view of our convention, $K(z, f) \equiv 1$ outside Ω. This, combined with Proposition 2.2, leads to uniform estimates

$$|h^\varepsilon(\xi) - h^\varepsilon(\zeta)|^2 \le \frac{\pi}{2\log(2/|\xi - \zeta|)} \int_{B_{r+3}} K(z, f)\, d\sigma(z) \tag{3.5}$$

for all ξ, $\zeta \in B_r$, $r > 1$, with $|\xi - \zeta| < 2$, and

$$d(h^\varepsilon(\xi), \infty) \le \frac{10}{1 + |\xi|}. \tag{3.6}$$

A consequence of (3.5) is that the homeomorphisms $h^\varepsilon\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ form an equicontinuous family on each compact subset of $\mathbb{C}$. By the Arzelà-Ascoli theorem it is possible to extract a sequence h^{ε_i}, $i = 1, 2, \dots$, $\varepsilon_i \searrow 0$, that converges c-uniformly to a mapping $h\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Actually, in view of (3.6), the mappings h^{ε_i} converge uniformly on the extended complex plane $\widehat{\mathbb{C}}$ with respect to the chordal metric. According to Lemma 1.4 the limit mapping $h\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is monotone. In particular, the set

$$\Omega' = h^{-1}(\Omega) \tag{3.7}$$

is a domain.

Other uniform estimates follow from (2.2) and (2.5), particularly,

$$\int_{B_{r+1}} |Dh^\varepsilon(\omega)|^2\, d\sigma(\omega) \le \int_{h(B_{r+1})} K(z, f^\varepsilon)\, d\sigma(z) \le \int_{B_{r+3}} K(z, f)\, d\sigma(z)$$

for all $r \ge 1$. This shows that h^{ε_i} converges weakly in $W^{1,2}(B_{r+1})$. Thus $h \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$.

Now we define the function $\varphi\colon \Omega' \to \mathbb{C}$ by the rule

$$\varphi(\omega) = f(h(\omega)) \quad \text{for } \omega \in \Omega'. \tag{3.8}$$

We want to prove that φ is holomorphic. To this end, fix an arbitrary open subset $U \subset \Omega'$, compactly contained in Ω'. Thus $h^{\varepsilon_i}(U) \subset \Omega$ for sufficiently small ε_i, and we can examine the mappings $\varphi^\varepsilon: U \to \mathbb{C}$ for $\varepsilon \in \{\varepsilon_1, \varepsilon_2, \dots\}$:

$$\varphi^\varepsilon(\omega) = f(h^\varepsilon(\omega)) \quad \text{for } \omega \in U. \tag{3.9}$$

Applying the chain rule (see Lemma 1.1) we find that $\varphi^\varepsilon \in W^{1,2}(U)$ and

$$\frac{\partial \varphi^\varepsilon}{\partial \bar\omega} = f_z h^\varepsilon_{\bar\omega} + f_{\bar z} \overline{h^\varepsilon_\omega}, \qquad \frac{\partial \varphi^\varepsilon}{\partial \omega} = f_z h^\varepsilon_\omega + f_{\bar z} \overline{h^\varepsilon_{\bar\omega}}.$$

Then, equations (3.1) and (3.4) imply

$$\frac{\partial \varphi^\varepsilon}{\partial \bar\omega} = (\mu(z) - \mu^\varepsilon(z)) f_z \overline{h^\varepsilon_\omega},$$
$$\frac{\partial \varphi^\varepsilon}{\partial \omega} = (1 - \mu(z)\overline{\mu^\varepsilon(z)}) f_z h^\varepsilon_\omega,$$

where $z = h^\varepsilon(\omega)$. It follows from the definition of $\mu^\varepsilon(z)$ (see formula (3.2)) that

$$|\mu(z) - \mu^\varepsilon(z)|^2 \le \varepsilon(1 - |\mu^\varepsilon(z)|^2),$$
$$|1 - \mu(z)\overline{\mu^\varepsilon(z)}|^2 \le 1 - |\mu^\varepsilon(z)|^2.$$

Notice, too, that $J(\omega, h^\varepsilon) = (1 - |\mu^\varepsilon|^2)|h^\varepsilon_\omega|^2$ (see (3.4)).

Now we use the change of variables according to Lemma 1.2 to obtain

$$\begin{aligned}\int_U |\varphi^\varepsilon_{\bar\omega}|^2 &\le \varepsilon \int_U J(\omega, h^\varepsilon)|f_z(h^\varepsilon(\omega))|^2 \, d\sigma(\omega) \\ &= \varepsilon \int_{h^\varepsilon(U)} |f_z(z)|^2 \, d\sigma(z) \le \varepsilon \int_\Omega |f_z(z)|^2 \, d\sigma(z).\end{aligned}$$

In much the same way we obtain the estimate

$$\int_U |\varphi^\varepsilon_\omega|^2 \, d\sigma(\omega) \le \int_\Omega |f_z(z)|^2 \, d\sigma(z).$$

These two estimates imply that the sequence $\varphi^{\varepsilon_i} = f(h^{\varepsilon_i})$ converges to the mapping $\varphi = f(h)$ not only pointwise (because f is continuous) but also weakly in $W^{1,2}(U)$. In conclusion, $\varphi \in W^{1,2}(U)$, and we have

$$\frac{\partial \varphi}{\partial \bar\omega} = 0, \qquad \int_U |\varphi_\omega|^2 \, d\sigma(\omega) \le \int_\Omega |f_z|^2 \, d\sigma(z).$$

By the Weyl lemma φ is holomorphic in U. Since U was an arbitrary compact subdomain of Ω', φ is holomorphic in Ω' as well. This also implies inequality (0.6). To derive inequality (0.5) we use Lemma 1.3

$$\begin{aligned}\int_U |Dh|^2 \, d\sigma(\omega) &\le \lim_{\varepsilon\to 0} \int_U |Dh^\varepsilon|^2 \, d\sigma(\omega) \le \lim_{\varepsilon\to 0} \int_{h^\varepsilon(U)} K(z, f^\varepsilon) \, d\sigma(z) \\ &\le \int_\Omega K(z, f) \, d\sigma(z).\end{aligned}$$

Of course, U can now be replaced by Ω', proving (0.5).

What remains is to show that $h: \Omega' \to \Omega$ is a homeomorphism, which is the same as to show that h is one-to-one.

Recall that our function $f \in W^{1,2}(\Omega)$ is actually continuous and nonconstant. For a given point $a \in \Omega$, its preimage $h^{-1}(a) \subset \Omega'$ is a continuum (compact connected set) because h is a monotone mapping. Clearly, the analytic function $\varphi = f \circ h$ is constant on $h^{-1}(a)$. Hence $h^{-1}(a)$ consists of a single point, because otherwise φ would be constant on the whole of Ω'; thus, f would be constant on Ω.

In conclusion, $h\colon \Omega' \to \Omega$ is a homeomorphism, and we have the factorization $f = \varphi \circ h^{-1}$.

This proves Theorem 1.

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