PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 118, Number 1, May 1993

EQUIVALENCE OF THE DEFINING SEQUENCES FOR ULTRADISTRIBUTIONS

SOON-YEONG CHUNG AND DOHAN KIM

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We prove that $M_p^* p! \sim M_p$. This result is a stronger version of the conjecture $M_p^* p! \supset M_p$ which has been given by Komatsu in *Ultradistributions I, Structure theorems and a characterization* (J. Fac. Sci. Univ. Tokyo Sect. IA **20** (1973), 25–105).

Let M_p be a sequence of positive numbers with $M_0 = 1$, satisfying the following two conditions :

(M.1) $M_p^2 \le M_{p-1} M_{p+1}$, $p = 1, 2, \dots$,

(M.3) $\sum_{k=p+1}^{\infty} M_{k-1}/M_k \le Hp\, M_p/M_{p+1}$, $p = 1, 2, \dots$, for some $H > 0$.

We say that two sequences M_p and N_p are equivalent and denote it by $M_p \sim N_p$ if there are constants $A, B > 0$, such that

$$A^p N_p \le M_p \le B^p N_p, \qquad p = 1, 2, \dots .$$

The above sequence M_p is used to define the various spaces of ultradifferentiable functions and ultradistributions. These spaces are invariant under the equivalence '$\sim$' (see Komatsu [1] for details).

The purpose of this note is to show that $M_p^* p! \sim M_p$ which refines Theorems 11.5 and 11.8 in Komatsu [1]. Here

$$M_p^* = \sup_{t>0} \frac{t^p}{\exp M^*(t)} \quad \text{and} \quad M^*(t) = \sup_{p \in \mathbf{N}} \log \frac{p!\, t^p}{M_p}.$$

Theorem. *Let M_p be a sequence as above. Then*

$$M_p^* \sim \frac{M_p}{p!}, \qquad p = 1, 2, \dots .$$

Proof. Let $m_p = M_p/M_{p-1}$ and $l_p = p/m_p + \sum_{k>p}^{\infty} 1/m_k$. Then the sequence l_p decreases because M_p satisfies (M.1). Since M_p satisfies (M.3), there is a constant $A \ge 1$ such that

$$\frac{m_p}{pA} \le \frac{1}{l_p} \le \frac{m_p}{p}, \qquad p = 1, 2, \dots .$$

Received by the editors July 24, 1991.

1991 *Mathematics Subject Classification.* Primary 46F05.

Key words and phrases. Defining function, ultradistribution, equivalent.

The first author was partially supported by the GARC-KOSEF, and the second author was partially supported by the Ministry of Education.

Let $n_p = pl_1/l_p$, $n_0 = 1$, and $N_p = \prod_{j=0}^{p} n_j$. Then N_p is equivalent to M_p and satisfies (M.1), (M.3). Moreover, the sequence $N_p/p!$ also satisfies (M.1). It follows from these facts and Gorny's theorem (Mandelbrojt [2]) that

$$\frac{N_p}{p!} = \sup_{t>0} \frac{t^p}{\exp N^*(t)}.$$

On the other hand, the equivalence $N_p \sim M_p$ implies that $N_p/p! \sim M_p/p!$ and there are constants $A, B > 0$ such that

$$N^*(At) \leq M^*(t) \leq N^*(Bt), \qquad t > 0.$$

Thus,

$$\begin{aligned}\left(\frac{1}{B}\right)^p \frac{N_p}{p!} &= \sup_{t>0} \frac{t^p}{\exp N^*(Bt)} \leq \sup_{t>0} \frac{t^p}{\exp M^*(t)} = M_p^* \\ &\leq \sup_{t>0} \frac{t^p}{\exp N^*(At)} = \left(\frac{1}{A}\right)^p \frac{N_p}{p!}, \qquad p \in \mathbf{N}.\end{aligned}$$

Therefore, it follows that

$$M_p^* \sim \frac{N_p}{p!} \sim \frac{M_p}{p!},$$

which completes the proof.

References

1. H. Komatsu, *Ultradistributions* I, *Structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo Sect. IA **20** (1973), 25–105.
2. S. Mandelbrojt, *Séries adhérentes régularisation des suites, applications*, Gauthier-Villars, Paris, 1952.

Department of Mathematics, Duksung Women's University, Seoul 132–714, Korea

Department of Mathematics, Seoul National University, Seoul 151–742, Korea