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ON COMPOSITIONS OF CONFORMAL IMMERSIONS

MARCOS DAJCZER AND ENALDO VERGASTA

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ABSTRACT. We consider conformal immersions of a manifold M^n, $n \geq 6$, into conformally flat manifolds. If the principal curvatures of $f\colon M^n \to N_{cf}^{n+1}$ have multiplicities at most $n-4$, we show that any $g\colon M^n \to \tilde{N}_{cf}^{n+2}$ can locally be written as $g = \rho \circ f$, where $\rho\colon N_{cf}^{n+1} \to \tilde{N}_{cf}^{n+2}$ is a conformal immersion.

1. INTRODUCTION

A classical result due to Cartan [Ca] states that a codimension one conformal immersion $f\colon M^n \to N_{cf}^{n+1}$ of an n-dimensional Riemannian manifold into a conformally flat Riemannian manifold is (locally) conformally rigid if $n \geq 5$ and the maximal multiplicity of the principal curvatures satisfies $\nu_f^c \leq n-3$ everywhere. Then any other conformal immersion $g\colon M^n \to \tilde{N}_{cf}^{n+1}$ is locally a composition $g = \rho \circ f$ for some local conformal diffeomorphism $\rho\colon N_{cf}^{n+1} \to \tilde{N}_{cf}^{n+1}$. Cartan's result was extended to codimension greater than one in [dCD].

For fixed $k \geq 2$, a natural problem is to find conditions on $f\colon M^n \to N_{cf}^{n+1}$ which imply that any conformal immersion $g\colon M^n \to \tilde{N}_{cf}^{n+k}$ is locally a conformal composition. That g is a *local conformal composition* means that, for each point $x \in M^n$, there exists a neighborhood $V \subset M$ of x and a conformal immersion $\rho\colon W \subset N_{cf}^{n+1} \to \tilde{N}_{cf}^{n+k}$ of an open subset of N_{cf}^{n+1} containing $f(V)$ such that $g = \rho \circ f$ along V. When $k = 2$, we prove the following result.

Theorem 1. *Let $f\colon M^n \to N_{cf}^{n+1}$ be a conformal immersion. Assume that $n \geq 6$ and $\nu_f^c(x) \leq n-4$ everywhere. If $g\colon M^n \to \tilde{N}_{cf}^{n+2}$ is a conformal immersion then there exists an open dense subset $\mathcal{U} \subset M$ such that, when restricted to $\mathcal{U}$, g is a local conformal composition.*

2. THE PROOF

We say that a submanifold $\overline{N}^{n+1} \subset \tilde{N}_{cf}^{n+2}$ is a *conformally flat hypersurface* if, with the metric induced by the inclusion map, $\overline{N}^{n+1}$ is conformally flat. Using Cartan's result, it is easy to check that Theorem 1 is equivalent to the following:

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Theorem 2. *Let* $f\colon M^n \to N^{n+1}_{cf}$ *be a conformal immersion. Assume that* $n \geq 6$ *and* $\nu^c_f(x) \leq n-4$ *everywhere. If* $g\colon M^n \to \tilde{N}^{n+2}_{cf}$ *is a conformal immersion then there exists an open dense subset* $\mathscr{U} \subset M^n$ *such that* $g|_{\mathscr{U}}$ *is locally contained in a conformally flat hypersurface of* $\tilde{N}^{n+2}_{cf}$.

To prove Theorem 2 we will make use of the following lemma on flat bilinear forms. We refer the reader to [dCD] or [Da] for notation, definitions, and some basic facts.

Lemma 3. *Let* $\beta\colon V \times V \to W^{k,2}$, $k \geq 3$, *be a nonzero symmetric bilinear form. Assume that* β *is flat and* $\dim N(\beta) < \dim V - \dim W$. *Then* W *admits an orthogonal direct sum decomposition* $W = W_1^{r,r} \oplus W_2^{k-r,2-r}$, *where* $r = 1$ *or* 2, *such that if* β_1 *and* β_2 *are the* W_1 *and* W_2 *components of* β, *respectively, then*

(i) β_1 *is nonzero and null,*
(ii) β_2 *is flat and* $\dim N(\beta_2) \geq \dim V - \dim W_2$.

Proof. Analogous to that of Lemma 2.2 in [dCD]. □

Proof of Theorem 2. We may assume that $N^{n+1}_{cf} = S^{n+1}$ is the unit Euclidean sphere, that $\tilde{N}^{n+2}_{cf} = \mathbf{R}^{n+2}$ is the flat Euclidean space, and that M^n is endowed with the metric induced by g. We consider S^{n+1} isometrically embedded in the light-cone $\mathbf{V}^{n+2}$ of the flat Lorentzian space $\mathbf{L}^{n+3}$ and contained in an $(n+2)$-dimensional affine hyperplane orthogonal to the axis of $\mathbf{V}^{n+2}$.

The map $F\colon M^n \to \mathbf{V}^{n+2} \subset \mathbf{L}^{n+3}$ defined by

$$F(x) = \frac{1}{\varphi(x)} f(x)$$

is an isometric immersion, where $\varphi\colon M^n \to \mathbf{R}$ is the positive function satisfying

$$\langle f_*(x)X, f_*(x)Y\rangle = \varphi^2(x)\langle X, Y\rangle$$

for any $X, Y \in T_xM$.

As in [dCD] or [Da], for a fixed point $x \in M^n$, the vector-valued second fundamental form $\alpha_F\colon TM \times TM \to T_FM^\perp$ of F in $\mathbf{L}^{n+3}$ is given by

$$\alpha_F = (\langle \alpha_F, \eta\rangle + \langle\ ,\ \rangle)\xi + \langle \alpha_F, \eta\rangle\eta + \alpha^*_F,$$

where the basis ξ, η for the orthogonal complement of $T_{f(x)}M^\perp$ into $T_{F(x)}M^\perp$ verifies

$$\langle \xi, \xi\rangle = 1, \qquad \langle \xi, \eta\rangle = 0, \qquad \langle \eta, \eta\rangle = -1$$

and $F(x) = \xi + \eta$. Here α^*_F is the $T_{f(x)}M^\perp$ component of α_F and satisfies

$$\alpha^*_F = \alpha_f/\varphi. \tag{1}$$

Now let

$$W = T_{g(x)}M^\perp \oplus \mathrm{Span}\{\xi\} \oplus \mathrm{Span}\{\eta\} \oplus T_{f(x)}M^\perp$$

be given the natural metric $\langle\langle\ ,\ \rangle\rangle$ of type $(3, 2)$. Define $\beta\colon T_xM \times T_xM \to W$ by

$$\beta = \alpha_g \oplus (\langle \alpha_F, \eta\rangle + \langle\ ,\ \rangle)\xi \oplus \langle \alpha_F, \eta\rangle\eta \oplus \alpha^*_F.$$

A straightforward computation shows that

$$\begin{aligned}\langle\langle\beta(X, Y), \beta(Z, W)\rangle\rangle &- \langle\langle\beta(X, W), \beta(Z, Y)\rangle\rangle \\ &= \langle\alpha_g(X, Y), \alpha_g(Z, W)\rangle - \langle\alpha_g(X, W), \alpha_g(Z, Y)\rangle \\ &\quad - \langle\alpha_F(X, Y), \alpha_F(Z, W)\rangle + \langle\alpha_F(X, W), \alpha_F(Z, Y)\rangle,\end{aligned}$$

and the Gauss equations for g and F imply that β is flat.

By definition of β, we have $\beta(X, X) \neq 0$ for $X \neq 0$; thus, $N(\beta) = 0$. By Lemma 3, $W = W_1 \oplus W_2$ decomposes orthogonally so that $\beta = \beta_1 \oplus \beta_2$, where

$$\beta_1\colon T_xM \times T_xM \to W_1^{r,r}, \qquad r \in \{1, 2\},$$

is nonzero and null and

$$\beta_2\colon T_xM \times T_xM \to W_2^{3-r,2-r}$$

is flat satisfying $\dim N(\beta_2) \geq n - 5 + 2r$.

We claim that $r = 2$. Assume $r = 1$. It follows that $\beta_1 = \phi\gamma$, where $\gamma \in W_1$ is a null vector and ϕ is a real-valued symmetric bilinear form. Thus there exists a unit vector $\delta \in T_{g(x)}M^\perp$ such that

$$\gamma = \cos\theta\delta + \sin\theta\xi + \cos\overline{\theta}\eta + \sin\overline{\theta}N,$$

where $N \in T_{f(x)}M^\perp$ is a unit vector. By definition, we have $Z \in N(\beta_2)$ if and only if $\beta(Z, X) = \beta_1(Z, X) = \phi(Z, X)\gamma$ for all $X \in T_xM$; therefore,

$$\langle\alpha_F(Z, X), \eta\rangle + \langle Z, X\rangle = \phi(Z, X)\sin\theta, \tag{2}$$

$$\langle\alpha_F(Z, X), \eta\rangle = \phi(Z, X)\cos\overline{\theta}, \tag{3}$$

and

$$\langle\alpha_F^*(Z, X), N\rangle = \phi(Z, X)\sin\overline{\theta} \tag{4}$$

for all $Z \in N(\beta_2)$ and $X \in T_xM$. From (2) and (3) we get

$$\phi(Z, X)(\sin\theta - \cos\overline{\theta}) = \langle Z, X\rangle, \tag{5}$$

which implies $\sin\theta - \cos\overline{\theta} \neq 0$. From (4) and (5) we obtain

$$\langle\alpha_F^*(Z, X), N\rangle = \frac{\sin\overline{\theta}}{\sin\theta - \cos\overline{\theta}}\langle Z, X\rangle. \tag{6}$$

Using (1), we conclude from (6) that f has a principal curvature with multiplicity at least $\dim N(\beta_2) \geq n - 3$. This is a contradiction and proves the claim.

Since $r = 2$, we have $\beta_1 = \phi_1\gamma_1 + \phi_2\gamma_2$, where ϕ_1, ϕ_2 are real-valued symmetric bilinear forms and γ_1, γ_2 are orthogonal null vectors. So we may write

$$\gamma_1 = \eta + \cos u\xi + \sin u\delta_1 \tag{7}$$

and

$$\gamma_2 = N + \cos v\xi + \sin v\delta_2, \tag{8}$$

where δ_1, δ_2 are unit vectors in $T_{g(x)}M^\perp$ verifying

$$\cos u\cos v + \sin u\sin v\langle\delta_1, \delta_2\rangle = 0.$$

Clearly, $\phi_1 = \langle \alpha_F , \eta \rangle$ and $\phi_2 = \langle \alpha_F^* , N \rangle$. Hence,

$$\beta_1 = \langle \alpha_F , \eta \rangle (\eta + \cos u \xi + \sin u \delta_1) + \langle \alpha_F^* , N \rangle (N + \cos v \xi + \sin v \delta_2) . \tag{9}$$

For any $Z \in N(\beta_2)$ and $X \in T_x M$, $\beta(Z , X) = \beta_1(Z , X)$ is equivalent to

$$\alpha_g(Z , X) = \langle \alpha_F(Z , X) , \eta \rangle \sin u \delta_1 + \langle \alpha_F^*(Z , X) , N \rangle \sin v \delta_2$$

and

$$\langle \alpha_F(Z , X) , \eta \rangle (1 - \cos u) + \langle Z , X \rangle = \langle \alpha_F^*(Z , X) , N \rangle \cos v .$$

Thus, from $\nu_f^c \le n - 4$, we have $1 - \cos u \ne 0$ and $\cos v \ne 0$; therefore,

$$\alpha_g(Z , X) = \langle \alpha_F(Z , X) , \eta \rangle (\sin u \delta_1 + \operatorname{tg} v (1 - \cos u) \delta_2) + \operatorname{tg} v \langle Z , X \rangle \delta_2 \tag{10}$$

and

$$\begin{aligned} \alpha_g(Z , X) = \langle \alpha_F^*(Z , X) , N \rangle \left(\frac{\sin u \cos v}{1 - \cos u} \delta_1 + \sin v \delta_2 \right) \\ - \frac{\sin u}{1 - \cos u} \langle Z , X \rangle \delta_1 . \end{aligned} \tag{11}$$

We easily conclude from (10) that g has a normal direction σ such that the tangent-valued second fundamental form A_σ in this direction has an eigenvalue with multiplicity at least $n - 1$ whose eigenspace contains $N(\beta_2)$.

From $r = 2$ we have that $\dim S(\beta) = 2, 3$. We claim that $\dim S(\beta) = 2$ if and only if σ is an umbilical direction. First observe that $\dim S(\beta) = 2$ if and only if $\beta_2 = 0$, and if $\beta_2 = 0$ then σ is umbilical by equation (10). Conversely, if $A_\sigma = cI$, consider the vector $\zeta = \sigma / c - \xi - \eta$. Then ζ is not null and $\langle\langle \beta , \zeta \rangle\rangle = 0$. This implies that $\dim S(\beta) = 2$ and proves the claim.

Assume that $\dim S(\beta) = 3$ on an open subset $V \subset M^n$. The 2-dimensional distribution $S(\beta) \cap S(\beta)^\perp$ is the (maximal) degeneracy space of the restriction of $\langle\langle \ , \ \rangle\rangle$ to the smooth distribution $S(\beta)$ and, therefore, is smooth. It follows easily that the vector fields δ_1 , δ_2 and the functions u, v in (7) and (8) can be taken to be smooth on V. The same conclusion holds on any open subset of M where $\dim S(\beta) = 2$.

Let $\mathscr{W} \subset M$ be the open subset of points where $\dim S(\beta) = 3$, and let $\mathscr{U}_1$ be the interior of $M \backslash \mathscr{W}$. Let σ be a smooth umbilical unit normal vector field defined on a connected component U_λ of $\mathscr{U}_1$. We claim that σ is parallel with respect to the normal connection of g. In fact, if σ is not parallel at $x \in U_\lambda$, we easily conclude from the Codazzi equation for A_σ that the second fundamental form $A_{\sigma^\perp}$ has a principal curvature with multiplicity at least $n-1$. The same holds in a neighborhood $W \subset U_\lambda$ of x, and it is a well-known fact that W must be conformally flat (cf. [CY]). By the classical Cartan-Schouten theorem for conformally flat hypersurfaces, we conclude that $\nu_f^c \ge n - 1$ on W, which is a contradiction and proves the claim. It follows from the claim that $g(U_\lambda)$ is contained in an umbilical hypersurface of $\mathbf{R}^{n+2}$.

For a connected component V_λ of $\mathscr{W}$, let σ be a smooth unit normal vector field such that the second fundamental form A_σ has eigenvalues μ , λ with multiplicities 1 and $(n-1)$, respectively. Set $\Delta = \ker(A_\sigma - \lambda I)$. We claim that λ is constant and σ is parallel along Δ. Consider orthonormal vector fields $Y_1 , \dots , Y_{n-1} \in \Delta$ such that $\nabla^\perp_{Y_j} \sigma = 0$ for $2 \le j \le n - 1$. From the

Codazzi equation for A_σ, Y_i, and Y_j for $2 \le i \ne j \le n-1$, we easily conclude that

$$Y_j(\lambda) = 0, \qquad 2 \le j \le n-2.$$

Now the Codazzi equation for A_σ, Y_1, and Y_j yields

$$\langle \nabla^\perp_{Y_1}\sigma, \sigma^\perp\rangle\langle A_{\sigma^\perp} Y_i, Y_j\rangle = 0, \qquad 1 \le i \ne j \le n-1, \tag{12}$$

and

$$Y_1(\lambda) = \langle \nabla^\perp_{Y_1}\sigma, \sigma^\perp\rangle\langle A_{\sigma^\perp} Y_j, Y_j\rangle, \qquad 2 \le j \le n-1. \tag{13}$$

If at some point $\langle \nabla^\perp_{Y_1}\sigma, \sigma^\perp\rangle \ne 0$, we obtain from (12) and (13) that $\mathrm{Span}\{Y_2, \dots, Y_{n-1}\}$ contains an $(n-3)$-dimensional umbilical subspace for g. Now (1) and (11) imply that $\nu_f^c(x) \ge n-3$, which is not possible, and this proves the claim.

Set $\mathscr{U} = \mathscr{U}_1 \cup \mathscr{U}_2 \cup \mathscr{U}_3$, where $\mathscr{U}_3 \subset \mathscr{W}$ is the open subset where $\lambda \ne 0$ and $\mathscr{U}_2$ is the interior of $\mathscr{W} \backslash \mathscr{U}_3$.

The image under g of any connected component of $\mathscr{U}_2$ is contained in a flat hypersurface of $\mathbf{R}^{n+2}$ by Proposition 3 of [DG]. If V_λ is a connected component of $\mathscr{U}_3$, define $c\colon V_\lambda \to \mathbf{R}^3$ by

$$c(x) = g(x) + r(x)\sigma(x), \qquad r(x) = 1/\lambda(x).$$

For all $Y \in \Delta$, we have

$$\tilde\nabla_Y c = Y - rA_\sigma Y = 0,$$

where $\tilde\nabla$ denotes the canonical connection of $\mathbf{R}^{n+2}$. If X is a unit tangent vector field orthogonal to Δ, we get

$$\tilde\nabla_X c = X - X(r)\sigma - rA_\sigma X - r\nabla^\perp_X\sigma.$$

In particular, since σ is not an umbilical direction, we have

$$\|\tilde\nabla_X c\|^2 > |X(r)|^2;$$

hence, from the curve c and the function r we can construct a conformally flat hypersurface in $\mathbf{R}^{n+2}$ as described in [dCDM] or [Da] which contains $g(V_\lambda)$. This concludes the proof. □

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IMPA, Est. Dona Castorina 110, 22460-320 Rio de Janeiro, Brazil
E-mail address: MARCOS@IMPA.BR

Institute of Mathematics, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40210 Salvador, Brazil