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ON THE HOMOLOGY OF POSTNIKOV FIBRES

Y. FÉLIX AND J. C. THOMAS

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ABSTRACT. Let k be a field of positive characteristic and X be a simply connected space of the homotopy type of a finite type CW complex. The Postnikov fibre $X_{[n]}$ of X is defined as the homotopy fibre of the n-equivalence $f_n: X \to X_n$ coming from the Postnikov tower $\{X_n\}$ of X. We prove that if the Lusternik-Schnirelmann category of X is finite, then $H_*(X_{[n]}; k)$ contains a free module on a subalgebra K of $H_*(\Omega X_n; k)$ such that $H_*(\Omega X_n; k)$ is a finite-dimensional free K-module.

Let k be a field of positive characteristic p and X be a simply connected space which has the homotopy type of a finite type CW complex. The Postnikov tower of X consists of a sequence of principal fibrations

$$X_n \xrightarrow{p_n} X_{n-1} \to K(\pi_n(X), n+1)$$

and of n-equivalences $f_n: X \to X_n$ satisfying $p_n f_n = f_{n-1}$. The homotopy fibre of f_n is then denoted by $X_{[n]}$ and is called the nth Postnikov fibre of X,

$$X_{[n]} \to X \xrightarrow{f_n} X_n .$$

The homotopy lifting property of the fibration f_n defines a natural action of $H_*(\Omega X_n; k)$ on $H_*(X_{[n]}; k)$ [8]. This action is called the *holonomy operation* and its behaviour in this context is the subject of this paper.

More generally we will consider a fibration of simply connected spaces

$$F \to E \xrightarrow{f} B$$

such that ΩB has a *stable r-stage Postnikov system* [7]. This means that ΩB can be obtained by a finite sequence of multiplicative fibrations

$$G_r \to G_{r-1} \to K_r, \quad r = 0, \dots, n, \qquad \Omega B = G_n, \qquad G_{-1} = \{*\},$$

with K_r a product of Eilenberg-Mac Lane spaces. The spaces ΩX_n are stable n-stable Postnikov systems. This happens also, for instance, when B has only a finite number of nonzero homotopy groups. We can now state our main theorems.

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The category of $f: E \to B$, $\operatorname{cat} f$, is defined as the least $m \leq \infty$ such that E can be covered by $m+1$ open sets U_i so that the restriction of f to each U_i is homotopic to zero [1]. Clearly $\operatorname{cat} f \leq \operatorname{cat} E$.

Theorem 1. *Let $F \to E \xrightarrow{f} B$ as before. We suppose that $\operatorname{cat} f < \infty$. Then*
(1) *there exists a nontrivial morphism of $H_*(\Omega B; k)$-modules*

$$H_*(F; k) \to H_*(\Omega B; k),$$

(2) *there exists a subalgebra K of $H_*(\Omega B; k)$ such that $H_*(F; k)$ contains a free K-module and such that $H_*(\Omega B; k)$ is a finite-dimensional free K-module.*

Theorem 2. *Let $F \to E \to B$ as before. Suppose that E has the homotopy type of a finite CW complex and ΩB is a product of Eilenberg-Mac Lane spaces. Then the algebra $H_*(\Omega B; k)$ is isomorphic to $K \otimes G$ with G finite-dimensional and $H_*(F; K)$ is a free K-module.*

One can remark that if f is homotopically trivial, then F has the homotopy type of the product $\Omega B \times E$, and Theorem 1 is obviously true in this case. The point is that the homology of the fibre F is never very far from being free; this is the content of the results.

In [3] Halperin and the authors establish a relation between $\operatorname{cat} f$ and some homological invariants of $H_*(F; k)$ as a module over $H_*(\Omega B; k)$. Let $G = \bigoplus_{i \geq 0} G_i$ be a graded Hopf algebra over the field k satisfying

$G_0 = k$;
$\dim G_i < \infty$ for any i;
G is cocommutative.

The *grade* of a graded G-module M is the greatest n (or ∞) such that $\operatorname{Ext}^n_G(M; G) \neq 0$. The *depth* of the Hopf algebra G is, by definition, the grade of the trivial module k. The main result of [3] reads as follows.

Theorem [3, Theorem A]. *With the above hypothesis,* $\operatorname{grade}(H_*(F; k)) \leq \operatorname{cat} f$.

Following Moore and Smith [7], we call a Hopf algebra G *p-solvable* if there exists a sequence of normal sub-Hopf algebras

$$k \subset G_{\langle -s \rangle} \subset G_{\langle -s+1 \rangle} \subset \cdots \subset G_{\langle 0 \rangle} = G$$

such that each quotient $G_{\langle t \rangle}//G_{\langle t-1 \rangle}$ is a commutative Hopf algebra with $x^p = 0$ for every x in $G_{\langle t \rangle}//G_{\langle t-1 \rangle}$, i.e., each quotient is a coprimitive Hopf algebra. The interest of p-solvable Hopf algebras in topology comes from the following result of Moore and Smith.

Theorem [7, Theorem 6.2]. *If ΩX is a stable r-stage Postnikov system, then the Hopf algebra $H_*(\Omega X; k)$ is p-solvable.*

Every finitely generated coprimitive Hopf algebra is finite dimensional. Thus from Lemma 1, every finitely generated p-solvable Hopf algebra is also finite dimensional. In particular, each element has finite height.

Lemma 1. *Let G be a finitely generated Hopf algebra and K be a normal sub-Hopf algebra such that the quotient $G//K$ is finite dimensional. Then K is also finitely generated.*

Proof. From the Hoschchild-Serre spectral sequence associated to the short exact sequence of Hopf algebras

$$K \to G \to G//K,$$

we obtain an isomorphism

$$\mathrm{Tor}^1_G(k, k) \cong \mathrm{Tor}^1_{G//K}(k, k) \oplus (\mathrm{Tor}^0_{G//K}(k, \mathrm{Tor}^1_K(k, k))/\mathrm{Im}(d_2)).$$

As $G//K$ is finite dimensional, the dimension of the vector space $\mathrm{Tor}^i_{G//K}(k, k)$ is finite for every i; therefore, $\mathrm{Tor}^1_G(k, k)$ is finite dimensional if and only if $\mathrm{Tor}^1_K(k, k)$ is finite dimensional. □

The next lemma is the main tool in the proof of Theorem 1. This is a generalization of [2, Proposition 3.1] with exactly the same proof.

Lemma 2. *Suppose* $0 \neq \omega \in \mathrm{Ext}^m_G(M, G)$, *$G$ a Hopf algebra, and M a G-module. Then for some finitely generated sub-Hopf algebra $K \subset G$, ω restricts to a nonzero element of* $\mathrm{Ext}^m_K(M, G)$.

Lemma 3. *A module M of finite grade on a p-solvable Hopf algebra G has grade zero.*

Proof. We denote by ω a nonzero element in $\mathrm{Ext}^m_G(M, G)$. It then results from Lemma 2 that, for some finitely generated sub-Hopf algebra $H \subset G$, ω restricts to a nonzero element in $\mathrm{Ext}^m_H(M, G)$; therefore, $\mathrm{Ext}^m_H(M, H) \neq 0$.

Write $M = \varinjlim M^\alpha$ with each M^α a finitely generated A-module. From the canonical isomorphisms

$$\mathrm{Ext}^s_H(M, H) = \varprojlim \mathrm{Ext}^s_H(M^\alpha, H),$$

one can see that if $\mathrm{Ext}^m_H(M, H) \neq 0$, then $\mathrm{Ext}^m_H(M^\alpha, H) \neq 0$ for some finitely generated submodule M^α.

The Hopf algebra H is p-solvable and finitely generated; therefore, H is finite dimensional and, hence, elliptic in the sense of [5]. By [4, Lemma 3.10] $m = 0$. □

Proof of Theorem 1. As the category of p is finite, the grade of the $H_*(\Omega B; k)$-module $H_*(F; k)$ is finite; therefore, by Lemma 3 the grade is zero. This implies the existence of a nontrivial morphism of $H_*(\Omega B; k)$-module $g: H_*(F; k) \to H_*(\Omega B; k)$.

For the sake of simplicity we denote $G = H_*(\Omega B; k)$ and $M = H_*(F; k)$. By hypothesis there exists a sequence of normal sub-Hopf algebras

$$k \subset A_{\langle -s\rangle} \subset A_{\langle -s+1\rangle} \subset \cdots \subset A_{\langle 0\rangle} = G$$

such that each quotient $A_{\langle t\rangle}//A_{\langle t-1\rangle}$ is isomorphic as an algebra to

$$A_{\langle t\rangle}//A_{\langle t-1\rangle} \cong \bigotimes_{i\in I_t} \Lambda x_i \otimes \bigotimes_{j\in J_t} k[y_j]/y_j^p.$$

For each degree q we denote by K_q the algebra generated by the x_i and the y_j of degree larger then q. By the previous decomposition, G is a free finitely generated K_q-module for each integer q.

Let m be an element of M such that $g(m) \neq 0$. The element $g(m)$ belongs to a finitely generated subalgebra H of G. Denote by t the maximal degree of the homogeneous elements of H. As G is a K_t free module, $K_t \cdot g(m) \cong K_t$. This implies that the K_t-module generated by m in M is free. □

Proof of Theorem 2. The homology Serre spectral sequence of the fibration

$$\Omega B \to F \to E$$

is a spectral sequence of $H_*(\Omega B; k)$-modules. As $H_*(E; k)$ is finite dimensional, there is only a finite number of nonzero differentials d_r in this spectral sequence. We will prove by induction on r that each E_r is a free finitely generated module over some subalgebra H_r of $G = H_*(\Omega B; k)$ such that G is a free finitely generated H_r-module.

This is true for E_2. The algebra $H_*(\Omega B; k)$ is a tensor product $\bigotimes_i \Lambda x_i \otimes \bigotimes_j k[y_j]/y_j^p$. The integer q will be defined by

$$q = (\text{the maximal degree of the homogeneous elements of } H_*(E; k)) + 2.$$

Denote by H_3 the tensor product of the components Λx_i and $k[y_j]/y_j^p$ with x_i and y_j of degree greater than q. G is then the tensor product $G = H_3 \otimes R_3$ with R_3 finite-dimensional. The E_2-term of the Serre spectral sequence, (E_2, d_2), is then isomorphic to $(H_3, 0) \otimes (H_*(E; k) \otimes R_3, d_2)$ as an H_3-module. Its homology E_3 is therefore isomorphic to $H_3 \otimes H(H_*(E; k) \otimes R_3, d_2)$ and is a free finitely generated H_3-module.

We proceed in exactly the same way for the general case. At each stage H_r will be the intersection of H_{r-1} and the tensor product of the components Λx_i and $k[y_j]/y_j^p$ with x_i and y_j of degree greater than $2 +$ (the maximal degree of the generators of E_{r-1} as an H_{r-1}-module). □

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Departmente Mathématique, Université Catholique de Louvain, Belgium

Departmente Mathématique, Université des Sciences et Techniques de Lille, France