PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 118, Number 1, May 1993

# HOMOTOPY, DIFFEOMORPHISM, AND DEFORMATION CLASSIFICATIONS OF CERTAIN SURFACES OF CLASS VII

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(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. We study the homotopy, diffeomorphism, and deformation classifications of all known surfaces of class VII which are not Hopf surfaces.

## 1. INTRODUCTION

Let $S_1$ and $S_2$ be two surfaces (all surfaces in this paper will be smooth compact complex surfaces). They are defined to be *deformation equivalent* if there exist connected complex spaces $\mathscr{X}$ and $T$ and a smooth proper holomorphic map $\Phi: \mathscr{X} \to T$, together with points $t_1, t_2 \in T$, such that $\Phi^{-1}(t_i) \cong S_i$. It is well known that if $S_1$ and $S_2$ are deformation equivalent, then they are diffeomorphic.

Let $\mathscr{E}'$ be the set of surfaces with nonnegative Kodaira dimensions and blow-ups of Hopf surfaces. One of the main results in [2] is that given any smooth and oriented 4-manifold $M^4$, there exist at most finitely many deformation equivalence classes of surfaces in $\mathscr{E}'$ which are diffeomorphic to $M^4$.

In this paper, we study the homotopy, diffeomorphism, and deformation classifications of those surfaces not included in $\mathscr{E}'$, that is, blow-ups of surfaces of class VII which are not Hopf surfaces. Due to the lack of a complete classification for surfaces of class VII, we concentrate on the set $\mathscr{E}$ of blow-ups of all *known* surfaces of class VII which are not Hopf surfaces. Our main result (Corollary 3.6) says that given any smooth and oriented 4-manifold $M^4$, there exist at most finitely many deformation equivalence classes of surfaces in $\mathscr{E}$ homotopic to $M^4$.

## 2. CLASSIFICATIONS OF INOUE SURFACES WITH $b_2 = 0$

### 2.1. Inoue surfaces with $b_2 = 0$ (see [3]).

(a) *Surfaces* $S_M^+$ *and* $S_M^-$. Let $M = (m_{i,j}) \in \mathrm{SL}(3, \mathbf{Z})$ with eigenvalues $\alpha$, $\beta$, and $\overline{\beta}$ such that $\alpha > 1$ and $\beta \neq \overline{\beta}$. Choose a real eigenvector $(a_1, a_2, a_3)$

Received by the editors August 23, 1991.
1991 *Mathematics Subject Classification*. Primary 32J15.
*Key words and phrases*. Deformation equivalent, surfaces of class VII, Hopf surfaces, Inoue surfaces, surfaces with global spherical shells.

of $\alpha$ and an eigenvector $(b_1, b_2, b_3)$ of $\beta$. Let $\mathbf{H}$ be the upper half of the complex plane. Define $G_M^+$ to be the group of analytic automorphisms of $\mathbf{H}\times\mathbf{C}$ generated by

$$g_0: (w, z) \to (\alpha w, \beta z),$$
$$g_i: (w, z) \to (w + a_i, z + b_i) \quad \text{for } i = 1, 2, 3.$$

Then the quotient $S_M^+ = \mathbf{H} \times \mathbf{C}/G_M^+$ is a surface of class VII.

Similarly, by using $\overline{\beta}$ and $(\overline{b}_1, \overline{b}_2, \overline{b}_3)$, one obtains $G_M^-$ and $S_M^-$.

(b) *Surfaces* $S^+_{N,p,q,r;t}$. Let $N = (n_{i,j}) \in \mathrm{SL}(2, \mathbf{Z})$ with two real eigenvalues $\alpha$ and $1/\alpha$ $(\alpha > 1)$. Choose real eigenvectors $(a_1, a_2)$ and $(b_1, b_2)$ of $\alpha$ and $1/\alpha$, respectively. Fix integers $p, q, r$ $(r \neq 0)$ and a complex number $t$. Define $(c_1, c_2)$ to be the solution of

$$(c_1, c_2) = (c_1, c_2) \cdot {}^tN + (e_1, e_2) + \frac{b_1a_2 - b_2a_1}{r} \cdot (p, q),$$

where $e_i = n_{i,1}n_{i,2}b_1a_2 + [n_{i,1}(n_{i,1} - 1)a_1b_1 + n_{i,2}(n_{i,2} - 1)a_2b_2]/2$ $(i = 1, 2)$. Let $G^+_{N,p,q,r;t}$ be the group of analytic automorphisms of $\mathbf{H} \times \mathbf{C}$ generated by

$$g_0^+: (w, z) \to (\alpha w, z + t),$$
$$g_i^+: (w, z) \to (w + a_i, z + b_iw + c_i) \quad \text{for } i = 1, 2,$$
$$g_3^+: (w, z) \to (w, z + (b_1a_2 - b_2a_1)/r).$$

Then the quotient $S^+_{N,p,q,r;t} = \mathbf{H} \times \mathbf{C}/G^+_{N,p,q,r;t}$ is a surface of class VII.

(c) *Surfaces* $S^-_{L,p,q,r}$. Let $L = (l_{i,j}) \in \mathrm{GL}(2, \mathbf{Z})$ with $\det(L) = -1$ having two real eigenvalues $\alpha$ and $-1/\alpha$ $(\alpha > 1)$. Choose real eigenvectors $(a_1, a_2)$ and $(b_1, b_2)$ of $\alpha$ and $-1/\alpha$, respectively. Fix integers $p, q, r$ $(r \neq 0)$, and define $(c_1, c_2)$ to be the solution of

$$-(c_1, c_2) = (c_1, c_2) \cdot {}^tL + (e_1, e_2) + \frac{b_1a_2 - b_2a_1}{r} \cdot (p, q),$$

where $e_i = l_{i,1}l_{i,2}b_1a_2 + [l_{i,1}(l_{i,1} - 1)a_1b_1 + l_{i,2}(l_{i,2} - 1)a_2b_2]/2$ $(i = 1, 2)$. Let $G^-_{L,p,q,r}$ be the group of analytic automorphisms of $\mathbf{H} \times \mathbf{C}$ generated by

$$g_0^-: (w, z) \to (\alpha w, -z),$$
$$g_i^-: (w, z) \to (w + a_i, z + b_iw + c_i) \quad \text{for } i = 1, 2,$$
$$g_3^-: (w, z) \to (w, z + (b_1a_2 - b_2a_1)/r).$$

Then the quotient $S^-_{L,p,q,r} = \mathbf{H} \times \mathbf{C}/G^-_{L,p,q,r}$ is a surface of class VII.

### 2.2. Classifications of Inoue surfaces with $b_2 = 0$.

**Lemma 2.1.** *Let* $M, N, L$ *be as in* (a), (b), *and* (c), *of* §2.1, *respectively. Then*

(i) $S_M^+$ *and* $S_M^-$ *are diffeomorphic, but not deformation equivalent*;
(ii) $G_M^+$, $G^+_{N,p,q,r;t}$, *and* $G^-_{L,p',q',r'}$ *are pairwisely nonisomorphic.*

*Proof.* (i) is proved by Inoue (see [7]).

(ii) From the definitions of $G_M^+$, $G^+_{N,p,q,r;t}$, and $G^-_{L,p',q',r'}$, we have

(a) $g_0g_ig_0^{-1} = g_1^{m_{i,1}}g_2^{m_{i,2}}g_3^{m_{i,3}}$ and $g_ig_j = g_jg_i$ for $i, j = 1, 2, 3$;

(b) $g_3^+$ commutes with $g_i^+$ for $i = 0, 1, 2$,
$g_0^+ g_1^+ (g_0^+)^{-1} = (g_1^+)^{n_{i,1}} (g_2^+)^{n_{1,2}} (g_3^+)^p$,
$g_0^+ g_2^+ (g_0^+)^{-1} = (g_1^+)^{n_{2,1}} (g_2^+)^{n_{2,2}} (g_3^+)^q$,
$(g_1^+)^{-1} (g_2^+)^{-1} g_1^+ g_2^+ = (g_3^+)^r$;
(c) $g_3^+$ commutes with $g_i^-$ for $i = 0, 1, 2$,
$g_0^- g_1^- (g_0^-)^{-1} = (g_1^-)^{l_{1,1}} (g_2^-)^{l_{1,2}} (g_3^-)^{p'}$,
$g_0^- g_2^- (g_0^-)^{-1} = (g_1^-)^{l_{2,1}} (g_2^-)^{l_{2,2}} (g_3^-)^{q'}$,
$(g_1^-)^{-1} (g_2^-)^{-1} g_1^- g_2^- = (g_3^-)^{r'}$.

For any group $G$, let $\lambda(G) = \{g \in G | g \bmod [G, G] \text{ is of finite order}\}$, and let $\mu(G) = G/[G, G]$. Then we obtain the following conclusions:

(d) $\lambda(G^+_{N,p,q,r;t}) = \langle g_1, g_2, g_3 \rangle \cong \mathbf{Z}^3$,
$\lambda(G^+_{N,p,q,r;t}) = \langle g_1^+, g_2^+, g_3^+ \rangle$,
$\lambda(G^-_{L,p',q',r'}) = \langle g_1^-, g_2^-, g_3^- \rangle$;

(e) $\mu\lambda(G^+_{N,p,q,r;t}) = \langle g_1, g_2, g_3 \rangle \cong \mathbf{Z}^3$,
$\mu\lambda(G^+_{N,p,q,r;t})/\text{Torsion} = \langle g_1^+, g_2^+ \rangle \cong \mathbf{Z}^2$,
$\mu\lambda(G^-_{L,p',q',r'})/\text{Torsion} = \langle g_1^-, g_2^- \rangle \cong \mathbf{Z}^2$.

By (e), $G^+_M$ cannot be isomorphic to either $G^+_{N,p,q,r;t}$ or $G^-_{L,p',q',r'}$.

Next we show that $G^+_{N,p,q,r;t}$ and $G^-_{L,p',q',r'}$ cannot be isomorphic. Assume the contrary, and let $\phi \colon G^+_{N,p,q,r;t} \to G^-_{L,p',q',r'}$ be an isomorphism. By conclusions (a)–(e), we notice the following:

(f) $G^+_{N,p,q,r;t}/\lambda(G^+_{N,p,q,r;t}) = \langle g_0^+ \rangle \cong \mathbf{Z}$,
$G^-_{L,p',q',r'}/\lambda(G^-_{L,p',q',r'}) = \langle g_0^- \rangle \cong \mathbf{Z}$;
(g) $\lambda^2(G^+_{N,p,q,r;t}) = \langle g_3^+ \rangle \cong \mathbf{Z}$
$\lambda^2(G^-_{L,p',q',r'}) = \langle g_3^- \rangle \cong \mathbf{Z}$.

By (f), (c), and (g), there exist integers $a_0 = \pm 1$, $a_1$, $a_2$ such that

$$\phi(g_0^+) = (g_0^-)^{a_0} (g_1^-)^{a_1} (g_2^-)^{a_2} \mod \lambda^2(G^-_{L,p',q',r'}).$$

By (e), there exists $A = (a_{i,j}) \in \mathrm{GL}(2, \mathbf{Z})$ with $\det(A) = \pm 1$ such that

$$\phi(g_i^+) = (g_1^-)^{a_{i,1}} (g_2^-)^{a_{i,2}} \mod \lambda^2(G^-_{L,p',q',r'}) \text{ for } i = 1, 2.$$

Applying $\phi$ to the second and third identities in (b) and using the second and third identities in (c), we conclude that $N \cdot A = A \cdot L^{a_0}$. Thus,

$$1 = \det(N) = \det(L^{a_0}) = \det(L) = -1,$$

a contradiction. Therefore $G^+_{N,p,q,r;t}$ and $G^-_{L,p',q',r'}$ are not isomorphic. □

**Lemma 2.2.** *Let $S$ be an Inoue surface with $b_2 = 0$.*

(i) *If $\pi_1(S)$ is isomorphic to $G^+_M$, then $S$ is either $S^+_M$ or $S^-_M$.*
(ii) *If $\pi_1(S)$ is isomorphic to $G^+_{N,p,q,r;t}$, then $S$ is $S^+_{N,p,q,r;t}$.*
(iii) *If $\pi_1(S)$ is isomorphic to $G^-_{L,p,q,r}$, then $S$ is $S^-_{L,p,q,r}$.*

*Proof.* (i) This is also proved by Inoue (see [7]).

(ii) By Lemma 2.1(ii), the Inoue surface $S$ must be of the form $S^+_{N',p',q',r';t'}$. From the arguments in §8 of [3], we see that the ordered pairs $(N', p', q', r', t')$ and $(N, p, q, r, t)$ must be equal. Thus, $S$ is $S^+_{N,p,q,r;t}$.

(iii) This follows from Lemma 2.1(ii) and the arguments in §9 of [3]. □

**Proposition 2.3.** *Let $S_1$ and $S_2$ be two Inoue surfaces with $b_2 = 0$. Then*

(i) *$S_1$ and $S_2$ are homotopic if and only if they are diffeomorphic;*
(ii) *$S_1$ and $S_2$ are deformation equivalent if and only if they are the same.*

*Proof.* (i) Assume that $S_1$ and $S_2$ are homotopic. Then $\pi_1(S_1) \cong \pi_1(S_2)$. If $\pi_1(S_i)$ is isomorphic to $G^+_{N,p,q,r;t}$, then both $S_1$ and $S_2$ are $S^+_{N,p,q,r;t}$ by Lemma 2.2(ii). If $\pi_1(S_i)$ is isomorphic to $G^-_{L,p',q',r'}$, then both $S_1$ and $S_2$ are $S^-_{L,p',q',r'}$ by Lemma 2.2(iii). If $\pi_1(S_i)$ is isomorphic to $G^+_M$, then $S_i$ $(i = 1, 2)$ is either $S^+_M$ or $S^-_M$ by Lemma 2.2(i). From Lemma 2.1(i), we see that $S^+_M$ and $S^-_M$ are diffeomorphic; therefore, $S_1$ and $S_2$ are diffeomorphic.

(ii) Assume $S_1$ and $S_2$ are deformation equivalent. Again, $\pi_1(S_1) \cong \pi_1(S_2)$. If $\pi_1(S_i)$ is isomorphic to $G^+_{N,p,q,r;t}$ (respectively, $G^-_{L,p',q',r'}$), then both $S_1$ and $S_2$ are $S^+_{N,p,q,r;t}$ (respectively, $S^-_{L,p',q',r'}$). If $\pi_1(S_i)$ is isomorphic to $G^+_M$, then $S_i$ $(i = 1, 2)$ is either $S^+_M$ or $S^-_M$. By Lemma 2.1(i), $S^+_M$ and $S^-_M$ are not deformation equivalent; therefore, both $S_1$ and $S_2$ are either $S^+_M$ or $S^-_M$. □

**Theorem 2.4.** *Let $\mathscr{E}_1$ be the set of blow-ups of Inoue surfaces with $b_2 = 0$. Then for a smooth and oriented 4-manifold $M^4$, there exist at most two surfaces in $\mathscr{E}_1$ which are homotopic to $M^4$.*

*Proof.* Assume that $\widetilde{S}$ is the $k$-fold blow-up of an Inoue surface $S$ with $b_2 = 0$ and that $\widetilde{S}$ is homotopic to $M^4$. Then $\pi_1(S) \cong \pi_1(\widetilde{S}) \cong \pi_1(M^4)$. By Lemma 2.2, there exist at most two choices for $S$. Since $\chi(\mathscr{O}_S) = \chi(\mathscr{O}_{\widetilde{S}}) = 0$, by the Riemann-Roch formula, $k = \chi(\widetilde{S}) = b_2(\widetilde{S}) = b_2(M^4)$, where $\chi(\widetilde{S})$ is the Euler number of $\widetilde{S}$; therefore, there exist at most two choices for $\widetilde{S}$ (note that there exist exactly two choices for $\widetilde{S}$ if and only if $\pi_1(M^4)$ is isomorphic to some $G^+_M$). □

## 3. Classifications of surfaces of class VII with GSS and $b_2 > 0$

**Definition 3.1** (see [6]). A subset $\Sigma$ of a compact complex surface $S$ is defined to be a *global spherical shell* (abbreviated GSS) if

(i) $\Sigma = \{(x, y) \in \mathbf{C}^2 | r < |x|^2 + |y|^2 < r'\}$ for two positive numbers $r$ and $r'$,
(ii) the complement $S - \Sigma$ is connected.

*Remark* 3.2. The only known minimal surfaces of class VII with $b_2 > 0$ are Inoue surfaces with $b_2 > 0$ (see [4, 5]) as well as the surfaces $S_{n,\alpha,t}$ where $n \in \mathbf{Z}^+$, $0 < |\alpha| < 1$, and $t \in \mathbf{C}^n$ (see [4, 6, 1]). All these surfaces contain GSS.

**Lemma 3.3** (see [6]). *If $S$ contains a GSS, then $S$ is deformation equivalent to the $b_2(S)$-fold blow-up of a primary Hopf surface.*

**Lemma 3.4.** *Let $S_1$ and $S_2$ be two surfaces of class* VII *with GSS. Then the following statements are equivalent.*

(i) *$S_1$ and $S_2$ are deformation equivalent.*

(ii) $S_1$ *and* $S_2$ *are diffeomorphic.*
(iii) $S_1$ *and* $S_2$ *are homotopic.*
(iv) $b_2(S_1)$ *and* $b_2(S_2)$ *are equal.*

*Proof*. We only need to show that (iv) implies (i). Let $b_2 = b_2(S_1) = b_2(S_2)$. By Lemma 3.3, $S_i$ $(i = 1, 2)$ is deformation equivalent to the $b_2$-fold blow-up of a primary Hopf surface. It is well known that all primary Hopf surfaces are deformation equivalent; therefore, $S_1$ and $S_2$ are deformation equivalent. □

**Theorem 3.5.** *Let* $\mathscr{E}_2$ *be the set of blow-ups of minimal surfaces of class* VII *with GSS and* $b_2 > 0$. *Then, for a smooth and oriented* 4*-manifold* $M^4$, *there exist at most finitely many deformation classes of surfaces in* $\mathscr{E}_2$ *homotopic to* $M^4$.

*Proof*. Assume that $\widetilde{S}$ is the $k$-fold blow-up of a minimal surface $S$ of class VII with GSS and $b_2(S) > 0$ and that $\widetilde{S}$ is homotopic to $M^4$. Then

$$k + b_2(S) = k + \chi(S) = \chi(\widetilde{S}) = \chi(M^4);$$

thus, there are at most $\chi(M^4)$ choices for $b_2(S)$. By Lemma 3.4, there are at most $\chi(M^4)$ deformation equivalence classes for $S$. Since $k \leq \chi(M^4)$, there are finitely many deformation equivalence classes for $\widetilde{S}$. □

**Corollary 3.6.** *Let* $\mathscr{E}$ *be the set of blow-ups of all known minimal surfaces of class* VII *which are not Hopf surfaces. Then, for a smooth and oriented* 4*-manifold* $M^4$, *there exist at most finitely many deformation equivalence classes of surfaces in* $\mathscr{E}$ *which are homotopic to* $M^4$.

*Proof*. The only known minimal surfaces of class VII which are not Hopf surfaces are Inoue surfaces (with $b_2 = 0$ and $b_2 > 0$) and the surfaces $S_{n,\alpha,t}$ (see [7]). By Remark 3.2, the conclusion follows from Theorems 2.4 and 3.5. □

## Acknowledgment

The author is very grateful to Robert Friedman and John Morgan for access to the preliminary version of their book [2].

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