PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 118, Number 2, June 1993

FAITHFUL REPRESENTATIONS OF CROSSED PRODUCTS BY ENDOMORPHISMS

SARAH BOYD, NAVIN KESWANI, AND IAIN RAEBURN

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Stacey has recently characterised the crossed product $A \times_\alpha \mathbf{N}$ of a C^*-algebra A by an endomorphism α as a C^*-algebra whose representations are given by covariant representations of the system (A, α). Following work of O'Donovan for automorphisms, we give conditions on a covariant representation (π, S) of (A, α) which ensure that the corresponding representation $\pi \times S$ of $A \times_\alpha \mathbf{N}$ is faithful. We then use this result to improve a theorem of Paschke on the simplicity of $A \times_\alpha \mathbf{N}$.

In Cuntz's fundamental article [2] on the Cuntz algebras $\mathcal{O}_n$, he viewed them as crossed products of a core UHF-algebra by an endomorphism, and used methods developed by O'Donovan [8] for ordinary crossed products to establish the uniqueness and simplicity of the $\mathcal{O}_n$. Later, Paschke [9] gave an elegant generalisation, in which he proved the key lemma by different methods, but used the same overall strategy. Cuntz did not explain what a crossed product by an endomorphism was—he argued by analogy—but subsequently suggested that it could be usefully defined as a corner in a certain ordinary crossed product [3, p. 101]. Recently, endomorphisms of C^*-algebras have appeared elsewhere (cf. [1, 4] and the references given there), and this led Stacey to give a modern description of their crossed products in terms of covariant representations and universal properties [12]. He also verified that the candidate proposed in [3] has the required property [12, 3.3].

Here we shall push Stacey's ideas a little further, and use them to clarify the proofs of simplicity given in [2, 9]. Our central theme is that, now we have the appropriate abstractly defined crossed product, there is an analogue of O'Donovan's theorem which tells us when a given representation is faithful, and that this result was implicitly one of the main ingredients in Cuntz's and Paschke's arguments. As evidence that this approach is useful, we have been able to eliminate an unnecessary hypothesis from Paschke's theorem: we do not need to assume that our endomorphism has a left inverse.

We begin with a discussion of Stacey's definition and main results, making some minor improvements that will be useful later. (We are only interested

Received by the editors September 21, 1991.
1991 *Mathematics Subject Classification*. Primary 46L55.
Key words and phrases. C^*-algebra, endomorphism, covariant representation, crossed product, simple C^*-algebra.

here in Stacey's "multiplicity-one crossed product," although covariant representations of higher multiplicity will briefly appear in §3.) In §2, we present our version of O'Donovan's result, and use it to establish our extension of Paschke's theorem. Our proof follows Paschke's in outline, although we do have to be careful in spots to avoid the extra hypothesis. In our last section, we discuss some examples involving shifts on infinite tensor products, which show that our theorem is indeed an improvement on Paschke's.

1. Covariant representations and crossed products

Let α be an endomorphism of a C^*-algebra A with 1. A covariant representation of (A, α) (of multiplicity one) is a pair (π, S), where π is a unital representation of A on H and S an isometry on H satisfying $\pi(\alpha(a)) = S\pi(a)S^*$ for $a \in A$. In [12], the crossed product of A by α (of multiplicity one) is a triple (B, i_A, t) in which B is a C^*-algebra with identity, $i_A : A \to B$ is a unital homomorphism, and t is an isometry in B satisfying

(a) $i_A(\alpha(a)) = t i_A(a) t^*$ for $a \in A$;

(b) for every covariant representation (π, S) of (A, α) on H, there is a unital representation $\pi \times S$ of B on H such that $(\pi \times S) \circ i_A = \pi$ and $\pi \times S(t) = S$;

(c) t and $\{i_A(a) : a \in A\}$ generate B.

Stacey proves that the system (A, α) has, up to isomorphism, exactly one crossed product, and we denote it $(A \times_\alpha \mathbf{N}, i_A, t)$.

We want to discuss variations on condition (c). First, we note that Stacey considers also nonunital A, for which the isometry t will lie in $M(B)$ rather than B; this forced him to insist that elements of the form $i_A(a)t^m(t^*)^n$ do lie in B. Secondly, it would be preferable to use in (c) a generating set which spans a dense subalgebra of the crossed product, as one does in the case of an automorphism (cf. [11, Definition 1(c)]). While the elements $i_A(a)t^m(t^*)^n$ used in [12] do not span a subalgebra unless $t^* i_A(A) t$ is contained in $i_A(A)$, one can give a slightly different list of elements that always do:

Lemma 1.1. *The span $\mathscr{B}$ of the set*

$$\{(t^*)^m i_A(a) t^n : a \in A,\ m, n \in \mathbf{N}\}$$

*is a dense *-subalgebra of $A \times_\alpha \mathbf{N}$.*

Proof. We only need to check that $\mathscr{B}$ is closed under multiplication. But the covariance of (i_A, t) (equation (a) above) implies that

$$\begin{aligned}(1.1)\qquad &\big((t^*)^k i_A(a) t^l\big)\big((t^*)^m i_A(b) t^n\big)\\ &\quad = \begin{cases} (t^*)^{k+m-l} i_A(\alpha^{m-l}(a)\alpha^m(1)b) t^n & \text{if } m \geq l, \\ (t^*)^k i_A(a\alpha^l(1)\alpha^{l-m}(b)) t^{n+l-m} & \text{if } m < l, \end{cases}\end{aligned}$$

which gives the result.

Remark. If α is an automorphism, one usually puts a *-algebra structure on the space $k(\mathbf{Z}, A)$ of finitely supported functions from $\mathbf{Z}$ to A, and defines $A \times_\alpha \mathbf{Z}$ to be the enveloping C^*-algebra. It is possible to do the same here: start with

$$k(\mathbf{N} \times \mathbf{N}, A) = \mathrm{sp}\{\delta_{m,n} a : m \geq 0, n \geq 0, a \in A\},$$

and define multiplication and involution by thinking of $\delta_{m,n}$ as $(t^*)^m i_A(a)t^n$ and, for example, carrying (1.1) over; one can verify directly (though painfully) that the resulting formulas do give a *-algebra structure for $k(\mathbf{N} \times \mathbf{N}, A)$. The representations of this *-algebra are given by the covariant representations (of multiplicity one), although there may not be enough of these to ensure the enveloping C^*-seminorm is actually a norm (see below). Thus defining the crossed product this way involves taking a quotient before completing, and the approach becomes a bit unwieldy.

As Stacey points out, a system (A, α) may have no covariant representations, and Cuntz's realisation of $A \times_\alpha \mathbf{N}$ as a corner in an ordinary crossed product [3] shows why. Define A_∞ to be the C^*-direct limit of the system $A \xrightarrow{\alpha} A \xrightarrow{\alpha} A \xrightarrow{\alpha} \cdots$, and write i_k for the canonical map of the kth copy of A into A_∞. The vertical arrows in the diagram

$$\begin{array}{ccccccccc} A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & \cdots & \longrightarrow & A_\infty \\ \alpha\downarrow & \overset{\mathrm{id}}{\nearrow} & \alpha\downarrow & \overset{\mathrm{id}}{\nearrow} & \alpha\downarrow & \overset{\mathrm{id}}{\nearrow} & & & \downarrow \\ A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & \cdots & \longrightarrow & A_\infty \end{array}$$

induce a homomorphism $\alpha_\infty : A_\infty \to A_\infty$, which is an automorphism because the NE arrows induce an inverse. Stacey's main result is:

Proposition 1.2 [12, 3.3]. *Let i_∞, $i_{\mathbf{Z}}$ denote the canonical embeddings of A_∞, $\mathbf{Z}$ in the crossed product $A_\infty \times_{\alpha_\infty} \mathbf{Z}$, and let $p = i_\infty(i_0(1))$. Then*

$$\big(p(A_\infty \times_{\alpha_\infty} \mathbf{Z})p,\ i_\infty \circ i_0,\ p i_{\mathbf{Z}}(1)p\big)$$

is a crossed product for (A, α).

Corollary 1.3 (cf. [12, 2.2]). *Let I be the kernel of $i_0 : A \to A_\infty$. Then*

$$I = \bigcap \{\ker \pi : (\pi, S) \text{ is a covariant representation of } (A, \alpha)\}.$$

In particular, if α is injective, there is a covariant representation (π, S) with π faithful.

Proof. The proposition implies that the covariant representations of (A, α) have the form $(\rho \circ i_\infty \circ i_0, \rho(p i_{\mathbf{Z}}(1)p))$, for some representation ρ of $p(A_\infty \times \mathbf{Z})p$. Thus I is certainly contained in the right-hand side. On the other hand, if (ρ, U) is a covariant representation of $(A_\infty, \mathbf{Z}, \alpha_\infty)$ with ρ faithful, then $\ker(\rho \times U) \circ i_\infty \circ i_0 = \ker \rho \circ i_0 = \ker i_0$. Since there is always such a representation (ρ, U) (for example, the representation induced from a faithful representation of A_∞), this gives the first part. It also gives the second, since if α is injective, so is i_0.

2. Faithful representations

Given a covariant representation (π, S) of (A, α), it is natural to ask when the corresponding representation $\pi \times S$ of $A \times_\alpha \mathbf{N}$ is faithful. For automorphisms, O'Donovan proved that the representation $\pi \times U$ of $A \times_\alpha \mathbf{Z}$ is faithful if π is faithful and the inequality $\|z(0)\| \leq \|\pi \times U(z)\|$ holds for all z in the dense subalgebra $k(\mathbf{Z}, A)$. Using Stacey's crossed product, we can prove a similar result for endomorphisms.

Proposition 2.1. *Suppose A is a C^*-algebra with 1, α is an endomorphism of A, and (π, S) is a covariant representation of (A, α) satisfying*

(1) *π is faithful, and*

(2) *for any finite subset $\{a_{m,n}\}$ of A,*

$$\left\|\sum_m (S^*)^m \pi(a_{m,m}) S^m\right\| \le \left\|\sum_{m,n} (S^*)^m \pi(a_{m,n}) S^n\right\|.$$

Then $\pi \times S$ is a faithful representation of $A \times_\alpha \mathbf{N}$.

To see where condition (2) comes from, we point out how the arguments of Cuntz and Paschke can be used to prove that a representation $\pi \times S$ is faithful. First one shows that any element X of the *-algebra generated by $\pi(A)$ and S can be written in the form

$$\sum_{k=1}^{M} (S^*)^k X_{-k} + X_0 + \sum_{k=1}^{N} X_k S^k,$$

where $X_k \in \pi(A)$, then proves $\|X_0\| \le \|X\|$ (cf. [2, Proposition 1.7; 9, Lemma 3(a)]), and finally applies O'Donovan's theorem (or an argument like that used to prove it) to deduce that $\pi \times S$ is faithful. Without requiring that $S^*\pi(A)S \subset \pi(A)$, one can only insist that the X_k are sums of elements of the form $(S^*)^m \pi(a) S^m$; our condition (2) replaces the inequality $\|X_0\| \le \|X\|$.

As in [9], the key observation in the proof of the proposition is that $A \times_\alpha \mathbf{N}$ carries a natural continuous action $\hat{\alpha}$ of $\mathbf{T}$ such that $\hat{\alpha}_z(i_A(a)) = i_A(a)$ and $\hat{\alpha}_z(t) = zt$. To see this, just verify that $(A \times_\alpha \mathbf{N}, i_A, zt)$ is also a crossed product for (A, α) and that the uniqueness of the crossed product gives the required automorphism $\hat{\alpha}_z$ of $A \times_\alpha \mathbf{N}$; $\hat{\alpha}$ is continuous because $i_A(A)$ and t span a dense subspace. If dz denotes normalised Haar measure on $\mathbf{T}$, then $\theta(a) = \int \alpha_z(a)\, dz$ defines a norm-decreasing projection θ of $A \times_\alpha \mathbf{N}$ onto the fixed-point algebra $(A \times_\alpha \mathbf{N})^{\hat{\alpha}}$, and since $\hat{\alpha}_z((t^*)^m a t^n) = z^{n-m}(t^*)^m a t^n$, we have

$$\theta\left(\sum_{m,n} (t^*)^m a_{m,n} t^n\right) = \sum_m (t^*)^m a_{m,m} t^m.$$

Thus, since both $\pi \times S$ and θ are continuous, the inequality (2) extends to

$$\|\pi \times S(\theta(b))\| \le \|\pi \times S(b)\| \quad \text{for all } b \in A \times_\alpha \mathbf{N},$$

and the proposition will follow from the lemma below. This lemma is absolutely standard: the idea of its proof goes back at least as far as [5].

Lemma 2.2. *Suppose B is a C^*-algebra and $\beta : G \to \operatorname{Aut} B$ is a strongly continuous action of a compact group G on B. If π is a representation of B satisfying*

(1) *π is faithful on the fixed-point algebra B^β, and*

(2) *$\|\pi\left(\int_G \beta_s(b)\, ds\right)\| \le \|\pi(b)\|$ for all $b \in B$,*

then π is faithful on B.

Proof. As above, $\theta(b) = \int \beta_s(b)\, ds$ is a projection of norm 1 onto B^β. If $\pi(b) = 0$, then $\pi(b^*b) = 0$, (2) implies $\pi\left(\int \beta_s(b^*b)\, ds\right) = 0$, and (1) forces $\int \beta_s(b^*b)\, ds = 0$. If ρ is a faithful representation of B, then

$$0 = \left(\rho\left(\int_G \beta_s(b^*b)\, ds\right)\xi | \eta\right) = \int_G (\rho(\beta_s(b))\xi | \rho(\beta_s(b))\eta)\, ds$$

for any $\xi, \eta \in H_\rho$ (see, e.g., [11, Lemma 7]). Thus $0 = \int \|\rho(\beta_s(b))\xi\|^2\,ds$ for any $\xi \in H_\rho$, which by continuity of the action implies $\|\rho(\beta_s(b))\xi\| = 0$ for all $s \in G$, $\xi \in H_\rho$, and $b = 0$. This completes the proof of the lemma and of Proposition 2.1.

We shall now use Proposition 2.1 to extend Paschke's theorem about endomorphisms of strongly amenable C^*-algebras; recall that this class includes all direct limits of type I algebras [6].

Theorem 2.3. *Let A be a strongly amenable C^*-algebra with 1, and α an endomorphism of A which satisfies $\alpha(1) \neq 1$ and which has no nontrivial invariant ideals. Then for any covariant representation (π, S) of (A, α), the representation $\pi \times S$ of $A \times_\alpha \mathbf{N}$ is faithful.*

Since we intend to use Proposition 2.1, we start by observing that π is faithful because $\ker\pi$ is an invariant ideal; thus we only need to verify condition (2) of that proposition. As $\alpha(1) \neq 1$ implies that $SS^* = \pi(\alpha(1)) \neq 1$, the pair $(\pi(A), S)$ satisfies all Paschke's hypotheses, except possibly $S^*\pi(A)S \subset \pi(A)$. He does not use this particular assumption in his Lemma 2, however, and we can borrow its proof.

Lemma 2.4. *There is a state f on $C^*(\pi(A), S)$ such that*
(a) *$f|_{\pi(A)}$ is faithful;*
(b) *$r = f(\pi(\alpha(1)))$ lies in $(0, 1)$;*
(c) *$f(\pi(a)b) = f(b\pi(a))$ for all $b \in C^*(\pi(A), S)$ and $a \in A$;*
(d) *$f(SbS^*) = rf(b)$ for all $b \in C^*(\pi(A), S)$;*
(e) *$f((S^*)^m\pi(a)S^n) = 0$ whenever $n \neq m$ and $a \in A$.*

Proof. Paschke's argument gives f (he calls it f_0) satisfying the first four conditions, and we only have to check (e). First suppose $n > m$. Then

$$\begin{aligned} f((S^*)^m\pi(a)S^n) &= r^{-n}f(S^n(S^*)^m\pi(a)S^n(S^*)^n) \quad \text{by (d)} \\ &= r^{-n}f(S^{n-m}\pi(\alpha^m(1)a\alpha^n(1))) \\ &= r^{-n}f(\pi(\alpha^m(1)a\alpha^n(1))S^{n-m}) \quad \text{by (c)} \\ &= r^{-n}f(S^m(S^*)^m\pi(a)S^n(S^*)^m(S^*)^{n-m}S^{n-m}) \\ &= r^{m-n}f((S^*)^m\pi(a)S^n) \quad \text{by (d).} \end{aligned}$$

Since $r \neq 1$, this implies $f((S^*)^m\pi(a)S^n) = 0$. Because $f(b^*) = \overline{f(b)}$, the case $m > n$ reduces to this one, and we have proved (e).

Corollary 2.5. *Let B be the *-subalgebra of $C^*(\pi(A), S)$ consisting of finite linear combinations of the elements $(S^*)^k\pi(a)S^k$. Then the state f of Lemma 2.4 is faithful on B.*

Proof. Suppose $b = \sum_{k=0}^K (S^*)^k\pi(a_k)S^k$ and $f(b^*b) = 0$. Then

$$\begin{aligned} S^K b(S^*)^K &= \sum_{k=0}^{K} S^{K-k}\pi(\alpha^k(1)a_k\alpha^k(1))(S^*)^{K-k} \\ &= \sum_{k=0}^{K} \pi\left(\alpha^K(1)\alpha^{K-k}(a_k)\alpha^K(1)\right) \end{aligned} \tag{2.1}$$

belongs to $\pi(A)$. Equation (d) implies that

$$f\big((S^K b(S^*)^K)^*(S^K b(S^*)^K)\big) = f\big(S^K b^* b(S^*)^K\big) = r^{-K} f(b^* b) = 0,$$

and hence $S^K b(S^*)^K = 0$ because f is faithful on $\pi(A)$. But

$$b = (S^*)^K \times (S^K b(S^*)^K) S^K,$$

so this forces $b = 0$ and gives the corollary.

We now verify condition (2) of Proposition 2.1. Let

$$X = \sum_{m,n=0}^{K} (S^*)^m \pi(a_{m,n}) S^n.$$

Following Paschke, we consider the GNS-representation (π_f, H_f, ξ_f) of $C^*(\pi(A), S)$ corresponding to the state f, and its compression to an appropriate subspace. We cannot use the obvious subspace $\overline{\pi_f(B)\xi_f}$, since we do not know if f is faithful on the C^*-algebra $\overline{B}$, and we use instead the span B_K of

$$\{(S^*)^k \pi(a) S^k : a \in A, k \le K\},$$

which we claim is a C^*-algebra. It follows from formula (1.1) that B_K is a *-subalgebra of $C^*(\pi(A), S)$, so we have to show it is closed. If $\{b_n\} \subset B_K$ and $b_n \to c$ in $C^*(\pi(A), S)$, then formula (2.1) shows that the sequence $\{S^K b_n (S^*)^K\}$ lies in $\pi(A)$. Since it is also Cauchy and the faithful representation π is isometric, there exists $a \in A$ such that $S^K b_n (S^*)^K \to \pi(a)$. But then $b_n = (S^*)^K S^K b_n (S^*)^K S^K \to (S^*)^K \pi(a) S^K$, and $c = (S^*)^K \pi(a) S^K$ lies in B_K. Thus B_K is closed, and hence a C^*-algebra.

Let P be the projection of H_f onto $\overline{\pi_f(B_K)\xi_f}$. Then

$$\begin{aligned}\|P\pi_f(X)P\| &= \sup\{|(\pi_f(X)\xi|\eta)| : \|\xi\| = \|\eta\| = 1,\ \ \xi, \eta \in PH_f\} \\ &= \sup\{|f(b_2^* X b_1)| : b_i \in B_K \text{ and } f(b_i^* b_i) = 1 \text{ for } i = 1, 2\}.\end{aligned}$$

It follows from (1.1) that $b_2^*(S^*)^m \pi(a) S^n b_1$ is the sum of terms of the form $(S^*)^k \pi(a_{k,l}) S^l$ with $k - l = m - n$, and hence by Lemma 2.4(e),

$$f(b_2^*(S^*)^m \pi(a) S^n b_1) = 0$$

unless $m = n$. Thus

$$f(b_2^* X b_1) = f\Big(b_2^*\Big(\sum_m (S^*)^m \pi(a_{m,m}) S^m\Big) b_1\Big),$$

and, if we write g for the restriction of f to B_K, then

$$\|P\pi_f(X)P\| = \Big\|P\pi_f\left(\sum (S^*)^m \pi(a_{m,m}) S^m\right) P\Big\| = \Big\|\pi_g\left(\sum (S^*)^m \pi(a_{m,m}) S^m\right)\Big\|$$

since $P\pi_f|_{B_K} P$ acting in $\overline{\pi_f(B_K)\xi_f}$ is unitarily equivalent to π_g. But $B_K \subset B$, so the corollary implies that π_g is faithful on B_K and we have

$$\begin{aligned}\Big\|\sum (S^*)^m \pi(a_{m,m}) S^m\Big\| &= \Big\|\pi_g\Big(\sum (S^*)^m \pi(a_{m,m}) S^m\Big)\Big\| \\ &= \|P\pi_f(X)P\| \le \|\pi_f(X)\| \le \Big\|\sum_{m,n} (S^*)^m \pi(a_{m,n}) S^n\Big\|.\end{aligned}$$

Thus Proposition 2.1 applies and $\pi \times S$ is faithful. This completes the proof of Theorem 2.3.

Corollary 2.6. *Let A be a strongly amenable C^*-algebra with 1, and let α be an injective endomorphism of A which has no nontrivial invariant ideals and satisfies $\alpha(1) \neq 1$. Then $A \times_\alpha \mathbf{N}$ is simple.*

Proof. The theorem says that every representation of $A \times_\alpha \mathbf{N}$ is faithful, so we only need to use Corollary 1.3 to obtain a covariant representation (π, S) with π nonzero.

Corollary 2.7 (cf. [9]). *Let A be a strongly amenable C^*-algebra with 1 acting on a Hilbert space H, and suppose S is a nonunitary isometry on H such that*

(a) *$SAS^* \subset A$,*

(b) *the only proper ideal J of A such that $SJS^* \subset J$ is $\{0\}$.*

Then $C^*(A, S)$ is simple.

Proof. Since S is nonunitary, $\alpha(a) = SaS^*$ defines an injective endomorphism of A with $\alpha(1) \neq 1$, and $\mathrm{id} \times S$ is an isomorphism of the simple algebra $A \times_\alpha \mathbf{N}$ onto $C^*(A, S)$.

3. Shifts on infinite tensor products

Let $\{A_n\}$ be an increasing sequence of subalgebras of a C^*-algebra with identity, such that $1 \in A_n$ for all n. For each n, let $B_n = \bigotimes_{k=1}^n A_k$, define $\phi_n : B_n \to B_{n+1}$ by

$$\phi_n(a_1 \otimes \cdots \otimes a_n) = a_1 \otimes \cdots \otimes a_n \otimes 1,$$

and let $A = \varinjlim (B_n, \phi_n)$ be the C^*-algebraic direct limit, as in [7, §6.1], with canonical embeddings $\phi^n : B_n \to A$. We fix a projection $e \in A_1$, and define $\alpha^n : B_n \to A$ by

$$\alpha^n(a_1 \otimes \cdots \otimes a_n) = \phi^{n+1}(e \otimes a_1 \otimes \cdots \otimes a_n);$$

we deduce from [7, 6.1.2] that the α^n induce an endomorphism $\alpha : A \to A$. It is customary to think of A as the infinite tensor product $A_1 \otimes \cdots \otimes A_n \otimes \cdots$; then

$$\phi^n(a_1 \otimes \cdots \otimes a_n) = a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes 1 \otimes \cdots ,$$
$$\alpha(a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \cdots) = e \otimes a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \cdots .$$

Example 3.1. Let $A_n = M_p(\mathbf{C})$ for all n, and let e be the rank-one projection e_{11}. Then A is the UHF-algebra UHF(p^∞) and the crossed product $A \times_\alpha \mathbf{N}$ is the Cuntz algebra $\mathcal{O}_n$. We give a brief proof of this well-known fact. If $\{S_i\}$ is a family of isometries satisfying $\sum S_i S_i^* = 1$ and μ is a multiindex, we write $S_\mu = S_{\mu_1} \cdots S_{\mu_n}$. If $e_{\mu\nu} = e_{\mu_1\nu_1} \otimes \cdots \otimes e_{\mu_n\nu_n}$, then $\psi^n(e_{\mu\nu}) = S_\mu S_\nu^*$ defines a compatible family of homomorphisms $\psi^n : B_n \to C^*(S_i)$, which induces a homomorphism ψ of $A = \varinjlim B_n$ onto the UHF core $\overline{\mathrm{sp}}\{S_\mu S_\nu^* : |\mu| = |\nu|\}$. Then $\psi(\alpha(\phi^n(e_{\mu\nu}))) = S_1(S_\mu S_\nu^*)S_1^*$, so (ψ, S_1) is covariant; because A is simple [7, 6.1.4], ψ is faithful and Theorem 2.3 implies that $\psi \times S_1$ is an isomorphism of $A \times_\alpha \mathbf{N}$ into $C^*(S_i)$. But $S_i = \psi(e_{i1} \otimes 1 \otimes \cdots)S_1$, and hence $\psi \times S_1$ maps $A \times_\alpha \mathbf{N}$ onto $C^*(S_i)$. (Indeed, $A \times_\alpha \mathbf{N}$ is generated by the isometries $t_i = i_A(e_{i1} \otimes 1 \otimes \cdots)t$ satisfying $\sum t_i t_i^* = 1$.)

In the previous example, the projection $e = e_{11}$ is rank-one, and it is easy to find covariant representations of multiplicity 1. If e has higher rank, it can be easier to write down covariant representations of higher multiplicity, and, therefore, it is of some interest to construct representations of multiplicity one from those of multiplicity n.

Example 3.2. Let $A_n = M_p(\mathbf{C})$ for all n, and let $e = \sum_{i\in I} e_{ii}$ have rank $|I|$. We fix a unit vector z in $\mathbf{C}^n$, and let $H = \bigotimes_{k=1}^{\infty} \mathbf{C}^p$; strictly speaking, H is the Hilbert space direct limit for the isometric embeddings

$$z_1 \otimes \cdots \otimes z_n \to z_1 \otimes \cdots \otimes z_n \otimes z,$$

but we think of H as the closed span of the tensors $z_1 \otimes \cdots \otimes z_n \otimes z \otimes z \otimes \cdots$. There is a natural representation π of A on H, induced by the representations $\pi^n : B_n \to B(H)$ such that

$$\begin{aligned}\pi^n(a_1 \otimes \cdots \otimes a_n)&(z_1 \otimes \cdots \otimes z_k \otimes z \cdots)\\ &= (a_1 z_1) \otimes \cdots \otimes (a_n z_n) \otimes \cdots \otimes z_k \otimes z \otimes \cdots\end{aligned}$$

for large k. If we define $S_i \in B(H)$ by

$$S_i(z_1 \otimes \cdots \otimes z_n \otimes z \otimes \cdots) = e_i \otimes z_1 \otimes \cdots \otimes z_n \otimes z \otimes \cdots,$$

then

$$S_i^*(z_1 \otimes \cdots \otimes z_n \otimes z \otimes \cdots) = (z_1|e_i) z_2 \otimes \cdots \otimes z_n \otimes z \otimes \cdots,$$

$\sum S_i S_i^* = 1$, and

$$\pi\big(\alpha(a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \cdots)\big) = \sum_{i\in I} S_i \pi(a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \cdots) S_i^*,$$

so that (π, S_i) is covariant of multiplicity $|I|$.

Lemma 3.3. *Suppose $\alpha \in \operatorname{End} A$ and (π, S_i) is a covariant representation of (A, α) on H of multiliplicity n. Then there is an isometry S on $H \otimes l^2$ such that*

(1) *$(\pi \otimes 1, S)$ is a covariant representation of (A, α) of multiplicity one;*
(2) *$S^*\big(\pi(A)\otimes 1\big)S \subset \pi(A)\otimes 1$ if and only if, for all $a \in A$, $S_i^*\pi(a)S_j = 0$ for $i \neq j$ and there exists $b \in A$ such that $S_i^*\pi(a)S_i = \pi(b)$ for $1 \le i \le n$.*

Proof. Let $\{T_i\}$ be a family of isometries on l^2 satisfying $\sum_i T_i T_i^* = 1$ and take $S = \sum_i S_i \otimes T_i^*$. Then

$$S^*S = \sum_{i,j} S_i^* S_j \otimes T_i T_j^* = \sum_{i,j} \delta_{ij} 1 \otimes T_i T_j^* = \sum_i 1 \otimes T_i T_i^* = 1 \otimes 1,$$

so S is an isometry, and

$$\begin{aligned}S(\pi(a) \otimes 1)S^* &= \sum_{i,j} S_i \pi(a) S_j^* \otimes T_i^* T_j = \sum_{i,j} S_i \pi(a) S_j^* \otimes \delta_{ij} 1\\ &= \left(\sum_i S_i \pi(a) S_i^*\right) \otimes 1 = \pi(\alpha(a)) \otimes 1,\end{aligned}$$

so (π, S) is covariant. Further,

$$S^*(\pi(a) \otimes 1)S = \sum_{i,j} S_i^* \pi(a) S_j \otimes T_i T_j^* = \pi(b) \otimes 1$$

if and only if

$$(3.1)\quad 1 \otimes T_k^* \left(\sum_{i,j} S_i^* \pi(a) S_j \otimes T_i T_j^* \right) 1 \otimes T_l = \pi(b) \otimes T_k^* T_l = \delta_{kl}(\pi(b) \otimes 1)$$

for all k and l. Since the left-hand side of (3.1) is $S_k^* \pi(a) S_l \otimes 1$, (2) follows easily.

Example 3.4. Suppose $A_n = M_p(\mathbf{C})$ for $n \geq 2$ and A_1 is a C^*-subalgebra of $M_p(\mathbf{C})$. If $e = \sum_{i \in I} e_{ii}$ as in §3.2, we can apply the lemma to the covariant representation constructed there. We compute

$$(S_i^* \pi(a_1 \otimes a_2 \otimes \cdots) S_j)(z_1 \otimes z_2 \otimes \cdots) = (a_1 e_j | e_i) a_2 z_1 \otimes a_3 z_2 \otimes \cdots .$$

Thus if A_1 is all of $M_p(\mathbf{C})$, $S_i^* \pi(a) S_j$ will not usually be zero and S will not satisfy $S^*(\pi(A) \otimes 1) S \subset \pi(A) \otimes 1$. By varying A, we can find examples where A is not simple, but α still has no invariant ideals. For example, take $p = 4$, $A_1 = M_2 \oplus M_2$, and $e = e_{11} + e_{33}$. Then we always have $S_1^* \pi(a) S_3 = 0$, but not necessarily $S_1^* \pi(a) S_1 = S_3^* \pi(a) S_3$, so $S^*(\pi(A) \otimes 1) S$ is not contained in $\pi(A) \otimes 1$. However, this A has two nontrivial ideals; neither contains any element of the form $e \otimes a_2 \otimes a_3 \otimes \cdots$, and hence cannot be invariant.

Acknowledgments

Our approach to crossed products by endomorphisms originated in the first author's honours thesis (University of New South Wales, 1989); we subsequently noticed the remarks at the start of [10, §4], which suggest that Paschke may have thought along similar lines much earlier. We thank Peter Stacey for helpful comments. This research was supported by the Australian Research Council.

References

1. W. B. Arveson, *Continuous analogues of Fock space*, Mem. Amer. Math. Soc., vol. 409, Amer. Math. Soc., Providence, RI, 1989.
2. J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173–185.
3. ——, *The internal structure of simple C^*-algebras*, Proc. Sympos. Pure Math., vol. 38, Amer. Math. Soc., Providence, RI, 1982, pp. 85–115.
4. S. Doplicher and J. E. Roberts, *Endomorphisms of C^*-algebras, cross products and duality for compact groups*, Ann. of Math. (2) **130** (1989), 75–119.
5. R. G. Douglas, *On the C^*-algebra of a one-parameter semigroup of isometries*, Acta Math. **128** (1972), 143–152.
6. B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc., vol. 127, Amer. Math. Soc., Providence, RI, 1972.
7. G. J. Murphy, *C^*-algebras and operator theory*, Academic Press, San Diego and London, 1990.
8. D. P. O'Donovan, *Weighted shifts and covariance algebras*, Trans. Amer. Math. Soc. **208** (1975), 1–25.
9. W. L. Paschke, *The crossed product by an endomorphism*, Proc. Amer. Math. Soc. **80** (1980), 113–118.
10. ——, *K-theory for actions of the circle group*, J. Operator Theory **6** (1981), 125–133.

11. I. Raeburn, *On crossed products and Takai duality*, Proc. Edinburgh Math. Soc. (2) **31** (1988), 321–330.

12. P. J. Stacey, *Crossed products of C^*-algebras by endomorphisms*, J. Austral. Math. Soc. (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEWCASTLE, NEW SOUTH WALES 2308, AUSTRALIA