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NOTE ON THE INTEGRABILITY OF SUPERHARMONIC FUNCTIONS

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ABSTRACT. Let D be a domain in $\mathbf{R}^n$ and let $S^+(D)$ be the set of all nonnegative superharmonic functions on D. It is shown that if $S^+(D) \subset L^p(D)$ with some $p > 0$, then for each $x_0 \in D$ there is a constant $C = C(D, p, x_0) > 0$ such that the inequality

$$\int_D u(x)^p dx \leq Cu(x_0)^p$$

holds for all $u \in S^+(D)$.

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Let D be a domain in $\mathbf{R}^n$, $n \geq 2$. Denote by $S^+(D)$ the set of all nonnegative superharmonic functions on D and by $L^p(D)$, $0 < p < \infty$, the space of pth integrable functions on D with respect to the n-dimensional Lebesgue measure dx. Our concern is the following problem: For what domains D does there exist a number $p > 0$ such that

$$S^+(D) \subset L^p(D) \tag{1}$$

holds? This global integrability of superharmonic functions was first studied by Armitage [1, 2] for a bounded domain with bounded curvature and then by Maeda and Suzuki [5] for Lipschitz domains. Recently, Masumoto [6] and Smith and Stegenga [7] discussed this property for a large class of plane domains and for Hölder domains, respectively. We remark that in order to show the inclusion (1), they all proved the following inequality: For each domain D under consideration, one can choose a number $p > 0$ satisfying

$$\int_D u(x)^p \, dx \leq Cu(x_0)^p \quad \forall u \in S^+(D) \tag{2}$$

with some constant $C = C(D, p, x_0) > 0$ ($x_0 \in D$: arbitrarily fixed). It is clear that (2) implies (1). In this article we shall prove the converse implication. Namely, we have the following

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Theorem. *Let D be a domain in $\mathbf{R}^n$ and $p > 0$. If $S^+(D) \subset L^p(D)$, then for each $x_0 \in D$ there is a constant $C = C(D, p, x_0)$ such that the inequality*

$$\int_D u(x)^p \, dx \leq Cu(x_0)^p \tag{3}$$

holds for all $u \in S^+(D)$.

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Proof of Theorem. Let $G(\cdot, \cdot)$ denote the Green function on D. We consider the functional Φ on the Banach space $L^1(D)$ defined by

$$\Phi(f) = \left(\int_D v_f(x)^p \, dx\right)^{1/p},$$

where

$$v_f(x) := \int_D \frac{G(x, y)}{G(x_0, y)} |f(y)| \, dy.$$

Then clearly $v_f \in S^+(D)$ so that $0 \leq \Phi(f) < \infty$. Furthermore, for any $a \in \mathbf{R}$, $\Phi(af) = |a|\Phi(f)$, $\Phi(f + g) \leq M_p(\Phi(f) + \Phi(g))$ with some constant $M_p > 0$, and $\liminf_{\|f-g\|_{L^1(D)} \to 0} \Phi(f) \geq \Phi(g)$. Thus the Gelfand theorem (cf. [3]) assures a constant $C > 0$ such that $\Phi(f) \leq C^{1/p} \|f\|_{L^1(D)}$. Since $\|f\|_{L^1(D)} = v_f(x_0)$, this constant C brings us to inequality (3) for each $u = v_f$, where $f \in L^1(D)$ and $f \geq 0$.

Now let v be a potential on the domain D with compact carrier, i.e., $v = \int_D G(\cdot, y) \, d\nu(y)$ and ν is a nonnegative Borel measure on D with compact support. To show (3) for v we may assume that $v(x_0) = \int_D G(x_0, y) \, d\nu(y) < \infty$. Then there is a sequence $\{f_n\}_{n=1}^{\infty}$ in $L^1(D)$ such that $f_n dy$ converges to $G(x_0, \cdot) d\nu$ vaguely as $n \to \infty$ and $\lim_{n\to\infty} \|f_n\|_{L^1(D)} = v(x_0)$. Since $\liminf_{n\to\infty} v_{f_n} \geq v$, $\int_D v(x)^p \, dx \leq \liminf_{n\to\infty} \int_D v_{f_n}(x)^p \, dx$ by Fatou's lemma. Hence inequality (3) for v with the constant C is also valid. Since every $u \in S^+(D)$ can be approximated by an increasing sequence of the said potentials v (cf. [4, Chapter 7, §3]), we obtain (3) for all $u \in S^+(D)$.

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Denote by $P(D)$ (resp. $H^+(D)$) the set of all potentials (resp. nonnegative harmonic functions) on D. The proof of theorem involves the following fact.

Corollary 1. *If $P(D) \subset L^p(D)$ with some $p > 0$, then $S^+(D) \subset L^p(D)$.*

Modifying the preceding argument, we have the following harmonic version of our theorem.

Corollary 2. *Let $p > 0$. If $L^p(D)$ contains $H^+(D)$, then for each $x_0 \in D$ there is a constant $C = C(D, p, x_0) > 0$ such that the inequality*

$$\int_D h(x)^p dx \leq Ch(x_0)^p \tag{4}$$

holds for all $h \in H^+(D)$.

Proof. We consider the Martin boundary X of D (cf. [4, p. 240]). Let $K(\cdot, \cdot)$ be the Martin kernel on $D \times X$ normalized at x_0, i.e., $K(x_0, \cdot) \equiv 1$. Taking a nonnegative Borel measure m on X with $\operatorname{supp}(m) = X$, we define the following functional Ψ on $L^1(X, m)$ (= the L^1 space on X with respect to m):

$$\Psi(f) = \left(\int_D \left(\int_X K(x, y)|f(y)|\,dm(y)\right)^p dx\right)^{1/p}.$$

Again, the Gelfand theorem gives a constant $C > 0$ such that

$$\Psi(f) \le C\|f\|_{L^1(X, m)} = \int_X K(x_0, y)|f(y)|\,dm(y). \tag{5}$$

Now let $h \in H^+(D)$. Then by the Martin representation theorem [4, p. 249], there is a nonnegative Borel measure μ on X such that $h = \int_X K(\cdot, y)\,d\mu(y)$. Approximating μ vaguely by a sequence $\{f_n dm\}_{n=1}^{\infty}$, where $f_n \in L^1(X, m)$ and $f_n \ge 0$, we have $h(x) = \lim_{n\to\infty} \int_X K(x, y) f_n(y) dm(y)$ and $h(x_0) = \lim_{n\to\infty} \int_X f_n(y)\,dm(y)$. Thus from (5), inequality (4) for h follows.

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