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CYCLIC INEQUALITIES

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Dedicated to Professor Tomiyama on his sixtieth birthday

ABSTRACT. We prove a class of cyclic inequalities including the ones that are related with the positivity of Choi's maps in matrix algebras.

INTRODUCTION

Inspired by Nowosad's results on Shapiro's cyclic inequalities, we shall give a proof of another kind of cyclic inequalities that have been conjectured with the relation to the positivity of Choi's maps

$$\sum_{j=1}^{n} \frac{x_j}{(n-m)x_j + x_{j+1} + \cdots + x_{j+m}} \leq 1, \tag{1}$$
$$x_1 > 0, \ldots, x_n > 0, \quad 1 \leq m \leq n-1,$$

where the x_j's are cyclically identified as $x_{j+n} = x_j$.

The inequality (1), which is proved for $m = 1$ in [6], for $m = n-2$ in [1] (the case $m = n-1$ is trivial), and then conjectured in the above form by Nakamura and recently proved for the case $(n, m) = (5, 2)$ in [4], now shows that the map $\tau_m : Mat_n(\mathbb{C}) \to Mat_n(\mathbb{C})$ defined by

$$\tau_m(X) = (n-m)\varepsilon(X) + \sum_{j=1}^{m} \varepsilon(C^j X C^{*j}) - X$$

is positive as pointed out in [6], where ε denotes the conditional expectation to the diagonal algebra and C the cyclic permutation. See [4, 5] for further information.

In this paper, we shall prove more general inequalities: For positive integers l, m, n and a positive real number s such that $s \geq n$ and $1 \leq m \leq n-l$, we have

$$\sum_{j=1}^{n} \frac{x_j}{(s-m)x_j + x_{j+l} + \cdots + x_{j+l+m-1}} \leq \frac{n}{s}, \qquad x_1 > 0, \ldots, x_n > 0, \tag{2}$$

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which is, again by the argument in [6], equivalent to the positivity of the map

$$X \mapsto (s-m)\varepsilon(X) + \sum_{j=l}^{l+m-1} \varepsilon(C^{*j}XC^j) - \frac{s}{n}X.$$

Preliminaries

Let A be a commutative C^*-algebra containing 1 and let φ be a positive linear functional on A. Set $A_h^\times = \{x \in A; x^* = x,\ x \text{ is invertible in } A\}$. For $a \in A_h^\times$, we denote by $[a]$ the set of elements, say x, in $A_h^\times$ such that x and x^{-1} are approximated in the norm topology of A by Laurent polynomials of a (i.e., polynomials of a and a^{-1}). Note that $[a]$ is a closed subgroup of $A_h^\times$ containing a.

The next theorem, due to Nowosad [3, Theorem 1.8], is essential in our proof of (2).

Theorem 1. *Let $T : A \to A$ be a bounded linear operator such that $T(x^*) = T(x)^*$ $\forall x \in A$. If the functional $\lambda_T : A_h^\times \ni x \mapsto \varphi(x^{-1}T(x))$ has local maximums at $x = 1$ and $x = a \in A_h^\times \setminus \mathbb{R}1$, then λ_T is constant on the subset $[a]$.*

We shall apply Theorem 1 to the case $A = \overbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}^{n \text{ times}}$ and $\varphi(x) = x_1 + \cdots + x_n$, $x = (x_1, \dots, x_n)$. Linear operators in A are identified with $n \times n$ matrices in an obvious way. Let $S = (s_{ij})$ be an $n \times n$ matrix with nonnegative real entries such that S and its transpose ${}^tS$ admit $\mathbf{1} = (1, \dots, 1)$ as an eigenvector. Then S and ${}^tS$ have a common eigenvalue $s > 0$ for the eigenvector $\mathbf{1}$ (we exclude the trivial case $S = 0$). Such a matrix S leaves the set $A_{++} \equiv \{(x_1, \dots, x_n) \in A; x_1 > 0, \dots, x_n > 0\}$ invariant, and we can consider a function f_S defined by

$$f_S(x) = \varphi(S(x)^{-1}x), \qquad x \in A_{++}.$$

Note that $f_S(x)$ depends only on the ray $\mathbb{R}_+x = \{\alpha x; \alpha > 0\}$, and hence f_S can be regarded as a function on the projective set $A_{++}/\mathbb{R}_+$.

Lemma 2. *$\mathbf{1} \in A_{++}$ is a critical point of f_S and its Hessian is given by $2\,{}^tSS - s(S + {}^tS)$.*

Proof. By a direct computation. □

Lemma 3. *Suppose that S is invertible. If the matrix $s(S+{}^tS) - 2\,{}^tSS$ is positive semidefinite and its kernel is spanned by $\mathbf{1}$, then $\mathbf{1}$ is a unique (up to scalar factor) point that gives a local maximum of f_S on A_{++}.*

Proof. By Lemma 2, f_S takes a local maximum at $\mathbf{1}$. To see the uniqueness, suppose on the contrary that there is another point $a \in A_{++} \setminus \mathbb{R}\mathbf{1}$ of local maximum. Then Theorem 1 with $T = S^{-1}$ tells us that the Hessian at $\mathbf{1}$ should be degenerate in a direction other than $\mathbf{1}$, which contradicts the assumption on the kernel of the Hessian. □

An $n \times n$ matrix S is called *cyclic* if it commutes with the cyclic permutation

$$C = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 0 & 0 \end{pmatrix}.$$

The following properties can be easily seen.

Lemma 4. (i) *S is cyclic if and only if S is a polynomial of C.*

(ii) *Eigenvectors of a cyclic matrix S are given by*

$$e_j = \begin{pmatrix} 1 \\ \zeta^j \\ \vdots \\ \zeta^{j(n-1)} \end{pmatrix}, \qquad j = 0, 1, \dots, n-1,$$

with $\zeta = e^{2\pi i/n}$. In fact, $Ce_j = \zeta^j e_j$.

Corollary 5. *A cyclic matrix S with nonnegative real entries fulfills the conditions in Lemma 3 if and only if its eigenvalues are contained in the set $\{z \in \mathbb{C};\ |z - s/2| < s/2\}$ except for s.*

Proof. Let $S = \sum_{j=0}^{n-1} s_j C^j$ with $s_j \geq 0$. Then the eigenvalue s of S for the eigenvector $\mathbf{1}$ is given by $s = \sum_{j=0}^{n-1} s_j$. Since eigenvalues of S are of the form $\lambda = \sum_{j=0}^{n-1} s_j \zeta^j$ with $\zeta^n = 1$, it is easy to see that s is a simple root of S. Thus the condition in Lemma 3 is equivalent to the positivity of

$$s(\lambda + \bar{\lambda}) - 2|\lambda|^2 = 2\left(\frac{s^2}{4} - |\lambda - \frac{s}{2}|^2\right)$$

for eigenvalues $\lambda \neq s$ of S, proving the assertion. □

Now we specialize to the case

$$S_{n,m,s} = (s-m)1 + C^l + C^{l+1} + \dots + C^{l+m-1}, \qquad 1 \leq m \leq n-l,\ \ n \leq s.$$

Lemma 6. *$S_{n,m,s}$ satisfies the spectral condition in Corollary 5.*

Proof. Since an eigenvalue λ_s of $S_{n,m,s}$ is of the form $\lambda + (s-n)$ with λ an eigenvalue of $S_{n,m,n}$, once the condition is proved for $s = n$, the general case follows:

$$\begin{aligned} |\lambda_s - s/2| &= |\lambda - n/2 + (s-n)/2| \\ &\leq |\lambda - n/2| + (s-n)/2 < n/2 + (s-n)/2 = s/2. \end{aligned}$$

So we concentrate on the case $s = n$.

Since the spectrum of C is $\{1, \zeta, \dots, \zeta^{n-1}\}$ with $\zeta = e^{2\pi i/n}$, eigenvalues of S are given by

$$\lambda_j = n - m + \zeta^{jl} + \zeta^{j(l+1)} + \dots + \zeta^{j(l+m-1)}, \qquad j = 0, 1, \dots, n-1.$$

We need to check that these eigenvalues (except for $\lambda_0 = n$) are contained in the open disk $|z - n/2| < n/2$, which can be seen as follows: For $m \leq n/2$,

$$\begin{aligned} |\lambda_j - n/2| &= |n/2 - m + \zeta^{jl} + \dots + \zeta^{j(l+m-1)}| \\ &< n/2 - m + |\zeta^{jl}| + \dots + |\zeta^{j(l+m-1)}| \\ &= n/2 - m + m = n/2, \end{aligned}$$

and for $m \geq n/2$,

$$\begin{aligned}|\lambda_j - n/2| &= |n/2 - m + \zeta^{jl} + \cdots + \zeta^{j(l+m-1)}| \\ &= |n/2 - m - \zeta^{j(l+m)} - \cdots - \zeta^{j(l+n-1)}| \\ &< m - n/2 + |\zeta^{j(l+m)}| + \cdots + |\zeta^{j(l+n-1)}| \\ &= m - n/2 + n - m = n/2. \quad \square\end{aligned}$$

Remark. The method of the proof shows that $S_{n,m,s}$ satisfies the spectral condition in Corollary 5 whenever $s > 2m$.

Proof of the inequality. First, we give the proof of (2) for $l = 1$ to get used to the reasoning. We express the summation in the left-hand side of (2) by $f_{n,m,s}(x)$. Note that $f_{n,m,s}(x) = f_S(x)$ for $S = S_{n,m,s}$. Recall that the function $f_{n,m,s}$ is homogeneous and can be regarded as a function on the subset $A_{++}/\mathbb{R}_+$ in the real projective space $\mathbb{P}^{n-1}$. By Lemmas 3 and 6, $f_{n,m,s}(x)$ has a unique (up to scalar factor) local maximum $f_{n,m,s}(\mathbf{1}) = n/s$ at $x = \mathbf{1}$. So to prove that this is the absolute maximum, we need to examine the behaviour of $f_{n,m,s}$ at the boundary ∂A_{++} of A_{++}, where the boundary ∂A_{++} is taken in the projective space:

$$\partial A_{++} = \{(x_1, \dots, x_n) \in A;\, x_1 \geq 0, \dots, x_n \geq 0,\ (x_1, \dots, x_n) \neq (0, \dots, 0)\}.$$

According to [3], we divide the boundary into two parts: $a \in \partial A_{++}$ is called in the regular boundary if the function $f_{n,m,s}(x)$ is analytically continued to a neighborhood of a, otherwise it is called in the singular boundary. So the problem is reduced to check (i) the behaviour of $f_{n,m,s}$ near the singular boundary and (ii) the values of $f_{n,m,s}$ at the regular boundary.

(i) Singular boundary. Let $a \in \partial A_{++}$ be in the singular boundary. Then the formal expression $f_{n,m,s}(a)$ contains at least one term of the form $0/0$. First we consider the case where only one term of indeterminate form appears in the summand of $f_{n,m,s}(a)$. By the cyclicity, we may assume that $a_n = a_1 = \cdots = a_m = 0$, $a_{m+1} > 0$, $a_{n-1} > 0$. When $x \in A_{++}$ approaches a, the sum of the first $n-1$ terms in the left-hand side of (2) approaches

$$\begin{aligned}&\frac{a_{m+1}}{(s-m)a_{m+1} + a_{m+2} + \cdots + a_{2m+1}} \\ &\quad + \cdots + \frac{a_{n-m}}{(s-m)a_{n-m} + a_{n-m+1} + \cdots + a_{n-1}} \\ &\quad + \cdots + \frac{a_{n-2}}{(s-m)a_{n-2} + a_{n-1}} + \frac{a_{n-1}}{(s-m)a_{n-1}},\end{aligned} \tag{3}$$

while the last term

$$\frac{x_n}{(s-m)x_n + x_1 + \cdots + x_m} \tag{4}$$

becomes a limit of indeterminate form.

Clearly, (3) is less than

$$\frac{a_{m+1}}{(s-m)a_{m+1}} + \cdots + \frac{a_{n-1}}{(s-m)a_{n-1}} = \frac{n-m-1}{s-m}$$

and (4) is less than

$$\frac{x_n}{(s-m)x_n} = \frac{1}{s-m}.$$

In total, they are less than

$$\frac{n-m}{s-m} \leq \frac{n}{s}.$$

Thus

$$\overline{\lim_{x \to a}} f_{n,m,s}(x) \leq \frac{n}{s}. \tag{5}$$

We can similarly manipulate the case where $f_{n,m,s}(a)$ contains more than one term of indeterminate form $0/0$ and show that (5) remains valid (notice the index j such that $a_j > 0$ and $a_{j-m-1} = a_{j-m} = \cdots = a_{j-1} = 0$).

(ii) Regular boundary. Let $a \in \partial A_{++}$ be in the regular boundary. We may assume that $a_n = 0$ without loss of generality. Then

$$\begin{aligned}
f_{n,m,s}(a) &= \frac{a_1}{(s-m)a_1 + a_2 + \cdots + a_{m+1}} \\
&\quad + \cdots + \frac{a_{n-m-1}}{(s-m)a_{n-m-1} + a_{n-m} + \cdots + a_{n-1}} \\
&\quad + \frac{a_{n-m}}{(s-m)a_{n-m} + a_{n-m+1} + \cdots + a_{n-1}} \\
&\quad + \cdots + \frac{a_{n-1}}{(s-m)a_{n-1} + a_1 + \cdots + a_{m-1}} \\
&\leq \frac{a_1}{(s-m)a_1 + a_2 + \cdots + a_m} \\
&\quad + \cdots + \frac{a_{n-m-1}}{(s-m)a_{n-m-1} + a_{n-m} + \cdots + a_{n-2}} \\
&\quad + \frac{a_{n-m}}{(s-m)a_{n-m} + a_{n-m+1} + \cdots + a_{n-1}} \\
&\quad + \cdots + \frac{a_{n-1}}{(s-m)a_{n-1} + a_1 + \cdots + a_{m-1}} \\
&= f_{n-1,m-1,s-1}(a_1, \ldots, a_{n-1}).
\end{aligned}$$

Thus the problem is reduced to the case $m = 1$ remarking that $(n-1)/(s-1) \leq n/s$. Now the estimate

$$\begin{aligned}
f_{n,1,s}(a) &= \frac{a_1}{(s-1)a_1 + a_2} + \cdots + \frac{a_{n-2}}{(s-1)a_{n-2} + a_{n-1}} + \frac{a_{n-1}}{(s-1)a_{n-1}} \\
&< \frac{a_1}{(s-1)a_1} + \cdots + \frac{a_{n-1}}{(s-1)a_{n-1}} = \frac{n-1}{s-1} \leq \frac{n}{s}
\end{aligned}$$

completes the analysis at the regular boundary, proving the inequality (2) for $l = 1$.

Remark. From the discussions at the singular boundary, we know that the inequality (2) with $l = 1$ breaks down for $s < n$. In fact, maximizing (3) and (4), respectively, we can find a sequence $x \in A_{++}$ such that

$$\overline{\lim_{x \to a}} f_{n,m,s}(x) = \frac{n-m}{s-m} > \frac{n}{s}.$$

This fact, combined with the remark after Lemma 6, shows that the spectral condition on the Hessian at $\mathbf{1}$ does not control the behaviour of the function $f_{n,m,s}$ at the boundary.

Generalization. Now the general case will be dealt with. Let us begin with the case $m = 1$. If n and l are relatively prime, then by changing the variables x_j's according to the permutation

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ l & 2l & 3l & \cdots & nl \end{pmatrix}$$

(the second line is calculated modulo n) the problem is reduced to the case $l = 1$. If n and l have the greatest common divisor $d \geq 2$, then the summand in $f_{n,m,s}(a)$ splits into d-parts and the problem is reduced to the case $d = 1$, just discussed.

So from now on, we assume that $m \geq 2$. Then most of the above argument is valid without changes; the only exception is in the discussion at the singular boundary, where we need to show that at least m terms become 0 in the summation of $f_{n,m,s}(a)$. Since $f_{n,m,s}(a)$ contains at least one term of indeterminate form $0/0$, we may assume that $a_1 = \cdots = a_m = 0$. We complete the analysis at the singular boundary by showing that the number of $1 \leq j \leq n$ satisfying

$$(*) \qquad a_j = 0, \qquad (a_{j+l}, a_{j+l+1}, \ldots, a_{j+l+m-1}) \neq (0, 0, \ldots, 0),$$

is $\geq m$. To see this, it is convenient to introduce the following terminology. A successive subsequence $a_p, a_{p+1}, \ldots, a_q$ of $\{a_j\}_{j \geq 1}$ is called a *null-string* if $a_p = a_{p+1} = \cdots = a_q = 0$ and $a_{p-1} > 0$, $a_{q+1} > 0$.

Consider a null-string of maximum length. By the cyclic nature, we may assume that it is of the form $\{a_1, \ldots, a_{k+m}\}$ $(k \geq 0)$. If

$$(a_{j+l}, a_{j+l+1}, \ldots, a_{j+l+m-1}) \neq (0, 0, \ldots, 0), \qquad k+1 \leq \forall j \leq k+m,$$

then the claim clearly holds. So we restrict ourselves to the case

$$(a_{j+l}, a_{j+l+1}, \ldots, a_{j+l+m-1}) = (0, 0, \ldots, 0),$$

for some j in $\{k+1, k+2, \ldots, k+m\}$.

Let j_2 be the maximum of such a j and $j_1 \leq j_2$ be such that

$$(**) \qquad a_{j_1+l}, \ldots, a_{j_2+l}, \ldots, a_{j_2+l+m-1} = (0, \ldots, 0)$$

and $a_{j_1+l-1} \neq 0$. Note that $j_1 \geq j_2 - k$ as the maximum length of null-strings is $k+m$. Since $(**)$ is a null string for $j_2 < k+m$, we see that the index j in the range

$$\max(1, j_1 - m) \leq j \leq j_1 - 1 \quad \text{or} \quad j_2 + 1 \leq j \leq k+m,$$

satisfies the condition $(*)$ and the number of j in this range is given by

$$\begin{aligned} &\min(j_1 - 1 + k + m - j_2, (j_1 - 1) - (j_1 - m - 1) + k + m - j_2) \\ &\quad = \min(j_1 + k - j_2 + m - 1, m + k + m - j_2) \\ &\quad \geq \min(m-1, m) = m - 1 \end{aligned}$$

because $j_1 \geq j_2 - k$ and $j_2 \leq k+m$.

Thus, if there exist two null-strings of length $k+m$, then the number N of j satisfying $(*)$ is estimated as

$$N \geq 2(m-1) \geq m$$

(note that $m \geq 2$).

Now suppose that $N = m-1$ and there is the unique null-string of maximum length. Since $N = m-1$, the above reasoning shows that $j_1 - m \leq 0$ and $j_1 + k - j_2 = 0$, which implies that the length of the null string containing $(**)$ is $\geq m+k$. Since the string of length $m+k$ is maximal and unique, we should have $j_1 + l = 1 \pmod{n}$, i.e., $j_1 = n+1-l$. Then $n+1-l \leq m$, which contradicts the assumption $m \leq n-l$. Thus we have completed the proof of (2).

Conjecture. Let n be a positive integer, s be a positive real number, and I be a subset of $\{1, \dots, n-1\}$. As a generalization of the inequality (2), it is natural to expect the inequality

$$\sum_{j=1}^{n} \frac{x_j}{(s-|I|)x_j + \sum_{i \in I} x_{i+j}} \leq \frac{n}{s},$$

where $|I|$ denotes the number of elements in I.

By a calculation based on the arguments in the proof of (2), we can check the validity of the above inequality in the most simple case $n = 4$, $I = \{1, 3\}$, which is not included in (2). The author has, however, no other clues on this problem.

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