PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 118, Number 2, June 1993

A MATRIX SOLUTION TO THE INVERSE PERRON-FROBENIUS PROBLEM

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(Communicated by George C. Papanicolaou)

ABSTRACT. Let f be a probability density function on the unit interval I. The inverse Perron-Frobenius problem involves determining a transformation $\tau\colon I \to I$ such that the one-dimensional dynamical system $x_{i+1} = \tau(x_i)$ has f as its unique invariant density function. A matrix method is developed that provides a simple relationship between τ and f, where f is any piecewise constant density function. The result is useful for modelling and predicting chaotic data.

1. INTRODUCTION

For many dynamical systems, only stochastic data in the form of an attractor $\mathscr{A}$ and a probability density function f supported on $\mathscr{A}$ is observed. The underlying dynamical system that generates the data is unknown. The problem of determining a deterministic transformation τ whose asymptotic dynamics (invariant measure) is given by f is referred to as the inverse Perron-Frobenius problem.

This problem was treated in [2], where a numerical algorithm is described that allows for the construction of a unimodal transformation τ on an interval of the real line and whose unique invariant density function is the observed density function f. Although the method is general for one-dimensional transformations, it is only a numerical algorithm requiring the computation of the integral

$$\varphi(x) = \int_x^1 f(y)\,dy$$

on the mesh $\{x_1, x_2, \dots, x_N\}$. As such the algorithm cannot provide a theoretical basis for finding a relationship between τ and f, which is the main objective of this note.

For density functions that are piecewise constant, the inverse problem was investigated in [1], where special classes of piecewise constant density functions were characterized as invariant densities for ergodic transformations. The class of densities considered in [1] is very restrictive, requiring the relative minima

Received by the editors April 30, 1991 and, in revised form, September 15, 1991.
1991 *Mathematics Subject Classification*. Primary 28D10; Secondary 58F11.
The authors' research was supported by NSERC and FCAR grants.

of f to have the value 0. The methods used are entirely graph-theoretic and do not lend themselves to generalization.

In §2 we present a complete generalization of the results of [2] using matrix analysis. This is accomplished by defining a new class of piecewise linear transformations called $\mathcal{P}$-semi-Markov, where $\mathcal{P}$ denotes a partition of the interval under consideration. The main result is a simple relationship between the density function and τ, where τ is a special transformation called a 3-band transformation.

We present applications in §3 and an example in §4.

2. Main result

In this section we introduce a class of piecewise linear transformations that, for a given defining partition, is richer than a class of Markov piecewise linear transformations but preserves its most important property: invariant densities are constant on elements of the defining partition. Let $\mathcal{P} = \{P_1, \dots, P_N\}$ be a partition of $I = [a, b]$ into intervals. We state the definition of a Markov transformation:

Definition 1. A transformation $\tau: I \to I$ is called *$\mathcal{P}$-Markov* if for any $i = 1, \dots, N$, $\tau|_{P_i}$ is monotonic and $\tau(P_i)$ is a union of intervals of $\mathcal{P}$. The following theorem is proved in [3].

Theorem 1. *If a transformation τ is $\mathcal{P}$-Markov and piecewise linear and expanding, i.e., for $i = 1, \dots, N$, $\tau|_{P_i}$ is linear with slope greater than 1, then any τ-invariant density is constant on intervals of $\mathcal{P}$.*

Now we define the class of $\mathcal{P}$-semi-Markov transformations.

Definition 2. A transformation $\tau: I \to I$ is called *$\mathcal{P}$-semi-Markov* if there exist disjoint intervals $Q_j^{(i)}$ such that for any $i = 1, \dots, N$ we have $P_i = \bigcup_{j=1}^{k(i)} Q_j^{(i)}$, $\tau|_{Q_j^{(i)}}$ is monotonic, and $\tau(Q_j^{(i)}) \in \mathcal{P}$.

It is easy to see that any $\mathcal{P}$-Markov transformation is $\mathcal{P}$-semi-Markov and that there are $\mathcal{P}$-semi-Markov transformations that are not $\mathcal{P}$-Markov.

The following theorem generalizes Theorem 1 to the case of semi-Markov transformations.

Theorem 2. *Let τ be a $\mathcal{P}$-semi-Markov transformation. Let $\tau|_{Q_j^{(i)}}$ be linear with slope greater than 1 for $j = 1, \dots, k(i)$, $i = 1, \dots, N$. Then any τ-invariant density is constant on intervals of $\mathcal{P}$.*

Proof. It is easy to see that τ is $\mathcal{Q}$-Markov, where $\mathcal{Q} = \{Q_j^{(i)}: 1 \le j \le k(i), 1 \le i \le N\}$. Let f be a τ-invariant density. By Theorem 1 f is constant on intervals $Q_j^{(i)}$. Let $f_j^{(i)}$ be the value of f on $Q_j^{(i)}$. Let us fix $1 \le i_0 \le N$, and let $1 \le j_1, j_2 \le k(i_0)$. The equations for the τ-invariant density give us

$$f_{j_1}^{(i_0)} = \sum_{(i,j)} |(\tau_j^{(i)})'|^{-1} f_j^{(i)}, \qquad f_{j_2}^{(i_0)} = \sum_{(i,j)} |(\tau_j^{(i)})'|^{-1} f_j^{(i)},$$

where $\tau_j^{(i)} = \tau|_{Q_j^{(i)}}$ and the sums are over all pairs (i, j) such that $\tau(Q_j^{(i)}) = P_{i_0}$. Since both sums on the right-hand side of equations are equal, $f_{j_1}^{(i_0)} = f_{j_2}^{(i_0)}$. $\square$

Definition 3. Let τ be a $\mathscr{P}$-semi-Markov piecewise linear transformation. We define the Perron-Frobenius matrix associated with τ by $\mathbb{M}_\tau = (a_{ij})_{1\le i,j\le N}$, where

$$a_{ij} = \begin{cases} |(\tau_k^{(i)})'|^{-1} & \text{if } \tau(Q_k^{(i)}) = P_j, \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbb{M}_\tau$ can be identified with the Perron-Frobenius operator $\mathbb{P}_\tau$ of τ, restricted to the space of functions constant on intervals of $\mathscr{P}$.

In the sequence of propositions and theorems below we prove that for any density f constant on intervals of $\mathscr{P}$ there exists a $\mathscr{P}$-semi-Markov transformation τ such that f is τ-invariant. The constructed transformation is piecewise linear and expanding and is related to f in a very simple way.

Proposition 1. *Let $\mathbb{P}$ be an $N\times N$ stochastic matrix. Let $\mathscr{R}$ be a partition of $I=[a,b]$ into N equal intervals. Then there exists an $\mathscr{R}$-semi-Markov transformation τ such that $\mathbb{M}_\tau = \mathbb{P}$.*

Proof. Let $\mathbb{P} = (p_{ij})_{1\le i,j\le N}$. Let $e_0^{(i)} = a + (i-1)(b-a)/N$, and let $R_i = [e_0^{(i)}, e_0^{(i+1)}]$, $i = 1, \dots, N$. Fix $1\le i\le N$. We will construct $\tau|_{R_i}$. Let $p_{ij_1}, \dots, p_{ij_k} > 0$ and $p_{ij_1} + \cdots + p_{ij_k} = 1$. Let

$$e_s^{(i)} = a + [(i-1) + p_{ij_1} + \cdots + p_{ij_s}](b-a)/N$$

for $s = 1, \dots, k$. We define $Q_s^{(i)} = [e_{s-1}^{(i)}, e_s^{(i)}]$ and $\tau|_{Q_s^{(i)}}(x) = \frac{1}{p_{ij_s}}(x - e_{s-1}^{(i)}) + e_0^{(j_s)}$. It easy to see that τ is an $\mathscr{R}$-semi-Markov, piecewise linear, and expanding (if all $p_{ij} < 1$) transformation and that $\mathbb{M}_\tau = \mathbb{P}$. □

Now we introduce a special class of $\mathscr{R}$-semi-Markov piecewise linear transformations.

Definition 4. An $\mathscr{R}$-semi-Markov piecewise linear transformation is said to be a 3-band transformation if its Perron-Frobenius matrix $\mathbb{M}_\tau = (p_{ij})$ satisfies: for any $1\le i\le N$, $p_{ij} = 0$ if $|i-j| > 1$.

The following theorem gives a simple formula satisfied by any invariant density of a 3-band transformation τ.

Theorem 3. *Let τ be a 3-band transformation with Perron-Frobenius matrix $\mathbb{M}_\tau = (p_{ij})$. Let f be any τ-invariant density and $f_i = f|_{R_i}$, $i = 1, \dots, N$. Then for any $2\le i\le N$ we have*

$$(*) \qquad p_{i,i-1}\cdot f_i = p_{i-1,i}\cdot f_{i-1}$$

Proof. The vector $f = (f_1, \dots, f_N)$ is a solution of the system of equations $f = f\mathbb{M}_\tau$. We will prove $(*)$ by induction. The first equation of the system is

$$f_1\cdot p_{1,1} + f_2\cdot p_{2,1} = f_1$$

or

$$p_{2,1}\cdot f_2 = p_{1,2}\cdot f_1,$$

so $(*)$ is true for $i=2$. Assume it is true for some $2\le i< N$; we will prove it is true for $i+1$. The ith equation of the system is

$$f_{i-1}\cdot p_{i-1,i} + f_i\cdot p_{i,i} + f_{i+1}\cdot p_{i+1,i} = f_i.$$

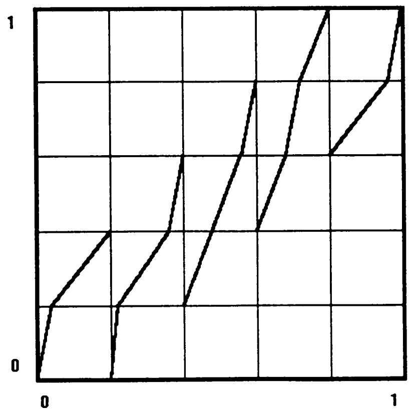

FIGURE 1

By the inductive hypothesis we have $f_{i-1} \cdot p_{i-1,i} = p_{i,i-1} \cdot f_i$. Substituting, we get $f_{i+1} \cdot p_{i+1,i} = f_i(1 - p_{i,i} - p_{i,i-1})$ or $f_{i+1} \cdot p_{i+1,i} = f_i \cdot p_{i,i+1}$, which proves the theorem. □

Example 1. Let $f = \frac{5}{16}(1, 8, 4, 2, 1)$ be a density constant on intervals of the partition of $[0, 1]$ into five equal parts. From Theorem 3 it follows that the 3-band matrix

$$\mathbb{M} = \begin{pmatrix} .2 & .8 & & & \\ .1 & .7 & .2 & & \\ & .4 & .4 & .2 & \\ & & .4 & .2 & .4 \\ & & & .8 & .2 \end{pmatrix}$$

corresponds to a 3-band piecewise expanding transformation τ, which preserves the density f. The graph of τ is presented in Figure 1.

Given a density f constant on intervals of the partition $\mathscr{R}$, it is not always possible to find an $\mathscr{R}$-Markov, piecewise linear, and expanding transformation τ that leaves it invariant. The simplest such case occurs when $\mathscr{R}$ consists of two equal intervals. Then the only possible invariant density is $(1, 1)$. Below we prove that we can always solve the problem using an $\mathscr{R}$-semi-Markov transformation.

Theorem 4. *Let $f = (f_1, \ldots, f_N)$ be a piecewise constant density on a partition $\mathscr{R}$ of $I = [a, b]$ into N equal intervals. Then there exists a 3-band piecewise expanding transformation τ such that f is τ-invariant.*

Proof. Let $g = [2 \cdot \max(f_i : i = 1, \ldots, N)]^{-1} f$. We define an $N \times N$ probability matrix, which by Proposition 1 is a Perron-Frobenius matrix of some 3-band transformation τ, as follows:

$$p_{11} = 1 - g_2, \quad p_{12} = g_2;$$

$$p_{i,i-1} = g_{i-1}, \quad p_{i,i} = 1 - g_{i-1} - g_{i+1}, \quad p_{i,i+1} = g_{i+1}, \qquad 2 \le i \le N-1;$$

$$p_{N,N-1} = g_{N-1}, \quad p_{N,N} = 1 - g_{N-1}.$$

It is obvious that τ is piecewise expanding. By Theorem 3, g and thus f is τ-invariant. □

It should be noticed that in general there exist infinitely many 3-band piecewise expanding transformations preserving a given density function.

Theorem 5. *Let $\mathscr{P} = \{P_1, \dots, P_N\}$ be a partition of $I = [a, b]$ into intervals and let the density $g = (g_1, \dots, g_N)$ be constant on intervals of $\mathscr{P}$. Then there exists a $\mathscr{P}$-semi-Markov piecewise linear and expanding transformation τ such that g is τ-invariant.*

Proof. Let $h\colon I \to I$ be defined as follows:

$$h|_{P_i}(x) = e_0^{(i)} + \frac{b-a}{N \cdot m(P_i)}(x - l(P_i)),$$

where $m(P_i)$ is the length of P_i and $l(P_i)$ is the left-hand side endpoint of P_i, $i = 1, \dots, N$. The function h is a piecewise linear homeomorphism and its Perron-Frobenius matrix is a diagonal matrix $H = \{[N \cdot m(P_i)]/(b-a)\}_{i=1}^N$. Let us define a function f piecewise constant on intervals of $\mathscr{R}$ (the partition of I into N equal intervals)

$$f = (f_1, \dots, f_N) = gH^{-1} = \left(\frac{g_i(b-a)}{N \cdot m(P_i)}\right)_{i=1}^N.$$

It is a density constant on intervals of $\mathscr{R}$. By Theorem 4, there exists a 3-band piecewise expanding transformation τ_0 such that f is τ_0-invariant, i.e., $f = f\mathbb{M}_{\tau_0}$. Let $\tau = h^{-1} \circ \tau_0 \circ h$. It is easy to see that τ is a $\mathscr{P}$-semi-Markov, piecewise linear, and expanding transformation. We have $\mathbb{M}_\tau = \mathbb{M}_{h^{-1}}\mathbb{M}_{\tau_0}\mathbb{M}_h = H^{-1}\mathbb{M}_{\tau_0}H$, so $g\mathbb{M}_\tau = g$ and g is τ-invariant. □

3. Applications

The 3-band transformations introduced in §2 are important because they provide an explicit and easy relationship between the invariant density of the transformation and the transformation itself. Given the density function, the construction of the transformation is not unique. Hence, if we want to ensure uniqueness of τ, we must make additional assumptions. To choose among many possible τ, we have to use some additional criteria, for example, Lyapunov exponents or other characteristics that can be extracted from the data.

The 3-band transformations have application to the following:

Chaotic data. Given chaotic experimental data, we form a histogram of the data and treat it as a density function. Then we can find a 3-band transformation τ preserving this density, and using orbits of τ, we can reproduce data statistically identical with the original data.

In [4] a nonlinear map is constructed directly from the data points themselves. The technique involves interpolating or approximating unknown functions from the data points and is extremely complex and computationally time consuming.

Random maps. Let τ_1, τ_2 be two transformations on the interval, say, and let $R = R(\tau_1, \tau_2, \alpha, \beta)$ denote the random map obtained from τ_1 and τ_2 by selecting τ_1 with probability α and τ_2 with probability β at any iteration of the process. (See [5] for detailed definitions.) The method of 3-band transformations applies to random maps just as easily as it does to ordinary maps. Given an invariant density f for a random map [5], we can construct a 3-band matrix that can be viewed as a point transformation and this transformation has f as its invariant density function. These ideas are relevant to dynamical system models for the two-slit experiment of quantum mechanics that will be presented in a subsequent note.

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