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AN ELEMENTARY SIMULTANEOUS APPROXIMATION THEOREM

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ABSTRACT. We will give an elementary and direct proof that for $f \in C^q[-1, 1]$ there exists a sequence of polynomials P_n of degree at most n $(n > 2q)$ such that for $k = 0, \dots, q$

$$|f^{(k)}(x) - P_n^{(k)}(x)| \le M_{q,k} \left(\frac{\sqrt{1-x^2}}{n}\right)^{q-k} E_{n-q}(f^{(q)}),$$

with $M_{q,k}$ depending only upon q and k. Moreover $f^{(q)}(\pm 1) = P_n^{(q)}(\pm 1)$.

1. INTRODUCTION

We will prove here the following

Theorem. *Let $f \in C^q[-1, 1]$. Then there exists a sequence of polynomials P_n of degree at most n $(n > 2q)$ such that for $k = 0, \dots, q$*

$$|f^{(k)}(x) - P_n^{(k)}(x)| \le M_{q,k} \left(\sqrt{1-x^2}/n\right)^{q-k} E_{n-q}(f^{(q)}),$$

where the constants $M_{q,k}$ depend only on q and k. Moreover, $f^{(q)}(\pm 1) = P_n^{(q)}(\pm 1)$.

Notation. Most of our notation is quite standard and will be introduced in context. As the first instance, we write $C^q[-1, 1]$ for the space of q times continuously differentiable functions, and

$$E_n(f) := \inf_{\text{degree } p_n \le n} \|f - p_n\|,$$

in which the norm is the usual supremum norm.

Context and antecedents. Those results that give pointwise estimates for polynomial approximation and/or establish sharp rates for simultaneous approximation of derivatives are often referred to as "of Timan type." The main feature of our theorem is a new and simply motivated proof directly employing basic results on approximation by trigonometric polynomials; its precise statement is also new. For comparison we state some of the better known previous results, introducing for brevity the notation

$$\Delta_n(x) := \frac{1}{n^2} + \frac{\sqrt{1-x^2}}{n}.$$

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Timan [11] showed the following:

> Let $f \in C^q[-1, 1]$. Then there is a sequence of polynomials P_n of degree at most n such that
> $$|f(x) - P_n(x)| \le M(\Delta_n(x))^q \omega(f^{(q)}; \Delta_n(x)).$$

The notation $\omega(f; h)$ (modulus of continuity) is defined by

$$\omega(f; h) := \sup_{|x-y|\le h} |f(x) - f(y)|.$$

Telyakovskiĭ [10] improved Timan's result by replacing $\Delta_n(x)$ with $\sqrt{1-x^2}/n$. Trigub [12] established the simultaneous approximation result that for $k = 0, \dots, q$

$$|f^{(k)}(x) - P_n^{(k)}(x)| \le M(\Delta_n(x))^{q-k}\omega(f^{(q)}; \Delta_n(x)).$$

Gopengauz [6] replaced $\Delta_n(x)$ (in both locations) in Trigub's result with $\sqrt{1-x^2}/n$, and Leviatan [8] replaced the modulus of continuity in Trigub's result with $E_{n-q}(f^{(q)})$. Our result combines features of the theorems of Gopengauz and Leviatan, and that of Leviatan follows in particular as a corollary from our theorem. Thus, the "interpolatory" result of Balázs, Kilgore, and Vértesi [2], useful in formulating estimates for simultaneous approximation via interpolation, also follows from the present theorem. More recently, the author stated and proved a version of this same theorem for the case $q = 1$ in Kilgore [7], which also surveys some recent applications to interpolation.

2. Method of proof

Our theorem will follow from two basic results on trigonometric best approximation of 2π-periodic functions (Lemma 1 and Lemma 2) and from several observations on derivatives (Lemmas 3, 4, 5). To avoid a convoluted presentation, we will not actually estimate the constants $M_{q,k}$, but it should be clear that reasonable estimates can be obtained, especially for small q. The estimates can be improved if the conclusion $f^{(q)}(\pm 1) = P_n^{(1)}(\pm 1)$ is dropped.

Before embarking on the sequence Lemmas 1–5, we introduce the notation

$$E_n^*(f) := \inf_{\text{order of } T_n \le n} \|f - T_n\|,$$

in which f is a 2π-periodic function and T_n is a trigonometric polynomial (linear combination of $1, \cos\theta, \dots, \cos n\theta, \sin\theta, \dots, \sin n\theta$). The order of T_n signifies the highest multiple of θ appearing in the actual expansion of T_n.

Lemma 1. *Let f be a k times differentiable periodic function. Then there exists for $k = 0, 1, 2, \dots$ a constant $\alpha_k \le (\pi/2)^2$ independent of f and n such that*

$$E_n^*(f) \le \alpha_k (n+1)^{-k} E_n^*(f^{(k)}).$$

Proof. This is a well-known result and is sufficient for our purposes. For a quite simple and basic proof one may consult Cheney [3]. In fact, $\alpha_k \le \pi/2$ for all k is known. The best possible values are estimated in Favard [5] or Achieser and Krein [1].

Our next lemma appeared in Czipszer and Freud [4], using as a proof the listed properties of the de la Vallée-Poussin means of the Fourier expansion, which are simply and succinctly demonstrated among other places in the text of Lorentz [9]. We prove the lemma for completeness and to highlight the independence of our theorem from the other results mentioned in the introduction.

Lemma 2. *Let f be a 2π-periodic function that is k times continuously differentiable. Let $T_n(\theta)$ be a trigonometric polynomial of order at most n satisfying for some constant C the inequality*

$$\|f(\theta) - T_n(\theta)\| \leq CE_n^*(f).$$

Then there exists a constant β_k independent of f and n such that

$$\|f^{(k)}(\theta) - T_n^{(k)}(\theta)\| \leq \beta_k CE_n^*(f^{(k)}).$$

Proof. Let $s_j f$ denote the Fourier expansion of f, truncated after order j. The de la Vallée-Poussin means are the linear operators $V_n f$ defined for $n = 1, 2, \ldots$ by

$$V_n f := \frac{1}{n} \sum_{j=n}^{2n-1} s_j f.$$

One has $V_n(f') = (V_n f)'$, $\|V_n\| \leq 3$ for $n = 1, 2, \ldots$, and $V_n(T_n) = T_n$ for order $T_n \leq n$. Immediately

$$\begin{aligned}
\|f' - T_n'\| &\leq \|f' - V_n(f')\| + \|(V_n f)' - T_n'\| \\
&\leq (1 + \|V_n\|)E_n^*(f') + 2n\|V_n(f) - T_n\| \\
&\leq 4E_n^*(f') + 2n\|V_n\| \, \|f - T_n\| \\
&\leq 4E_n^*(f') + 6CnE_n^*(f) \leq 4E_n^*(f') + 3C\pi E_n^*(f'),
\end{aligned}$$

using the listed properties of V_n, the Bernstein inequality, and Lemma 1. The lemma is thus demonstrated for $k = 1$ and can obviously be established by repetition for other values of k.

Another estimate for the constants β_k is found in the cited article of Czipszer and Freud, of the $\beta_k \leq A \log[\min(n, k) + 1]$, where the constant A is independent of n, f, and k.

Lemma 3. *Let $h(\theta)$ be a 2π-periodic function that is q times continuously differentiable. Then for each $n > q$ there exists a trigonometric polynomial $T_n(\theta)$ of order at most n and constants $\gamma_0, \ldots, \gamma_q$ independent of h and n such that for $k = 0, \ldots, q$*

(a) $\|h^{(k)}(\theta) - T_n^{(k)}(\theta)\| \leq (\gamma_k / n^{q-k}) E_n^*(h^{(q)})$,

(b) $h^{(k)}(0) - T_n^{(k)}(0) = h^{(k)}(\pi) - T_n^{(k)}(\pi) = 0$,

and T_n is even if h is even, odd if h is odd.

Proof. For $q > 0$ let the integer m be chosen as the greatest integer in $(n-1)/q$. Let $B_n(\theta)$ be a best approximation for h. Then (a) holds by Lemma 2 combined with Lemma 1.

We now define polynomials $T_{n,0}(\theta), \ldots, T_{n,q}(\theta)$ by the following procedure:

$$T_{n,0}(\theta) := B_n(\theta) + (h(0) - B_n(0)) \left(\frac{1 + \cos\theta}{2}\right) + (h(\pi) - B_n(\pi)) \left(\frac{1 - \cos\theta}{2}\right),$$

and we note that $T_{n,0}(\theta)$ satisfies conclusion (a) with appropriate constants $\gamma_k^{(0)}$ for $k = 0, \dots, q$. Moreover, $T_{n,0}$ satisfies (b) for the value $k = 0$. We now define $T_{n,k}$ for $k = 1, \dots, q$ in recursive fashion by

$$T_{n,k}(\theta) := T_{n,k-1}(\theta) + \left((h^{(k)}(0) - T_{n,k-1}^{(k)}(0) \left(\frac{1+\cos\theta}{2}\right) \right.$$
$$\left. +(-1)^{km}(h^{(k)}(\pi) - T_{n,k-1}^{(k)}(\pi)) \left(\frac{1-\cos\theta}{2}\right) \right) \frac{\sin^k m\theta}{k!m^k}.$$

We note first of all that $T_{n,k}$ satisfies (b) for derivatives from 0 up to k, and thus $T_{n,q}$ satisfies (b) of the lemma for derivatives 0 to q. We will define $T_n(\theta) := T_{n,q}(\theta)$. Lemma 2 suffices for the estimates required in (a), providing for each $k = 1, \dots, q$ a set of constants $\gamma_0^{(k)}, \dots, \gamma_q^{(k)}$ such that

$$\|h^{(j)}(\theta) - T_{n,k}^{(j)}(\theta)\| \le (\gamma_j^{(k)}/n^{q-j})E_n^*(h^{(q)}), \qquad j = 0, \dots, q.$$

To establish that $T_n(\theta)$ may be chosen even if $h(\theta)$ is even and odd if $h(\theta)$ is odd, it suffices to note that $B_n(\theta)$ may be chosen even if h is even or odd if h is odd, and the construction of the polynomials $T_{n,k}$ preserves evenness or oddness. Moreover, if h is odd, then $T_{n,0} = B_n$; $T_{n,2} = T_{n,1}$; $T_{n,4} = T_{n,3}$, and so on, and if h is even, then $T_{n,1} = T_{n,0}$; $T_{n,3} = T_{n,2}$; $T_{n,5} = T_{n,4}$, and so on up to $T_{n,q}$.

Lemma 4. *Let $\{H_n(\theta)\}$ be a sequence of q times continuously differentiable 2π-periodic functions satisfying*

$$H_n(\theta)/\sin^q\theta \to 0 \quad \text{as } \theta \to 0 \text{ or } \theta \to \pi$$

and satisfying for some sequence of nonnegative numbers $\{\varepsilon_n\}$ the inequalities $\|H_n^{(k)}(\theta)\| \le C/n^{q-k}\varepsilon_n$ for $k = 0, \dots, q$. Then for $k = 0, \dots, q$ the functions $H_{n,k}(\theta)$ satisfy $\|H_{n,k}(\theta)\| \le CK/n^{q-k}\varepsilon_n$ with K dependent only on q (and k), where $H_{n,0} := H_n$, and for $k = 0, \dots, q-1$

$$H_{n,k+1}(\theta) := H'_{n,k}(\theta) + (q-k)(\cos\theta H_{n,k}(\theta))/\sin\theta.$$

Proof. We will use induction and suppose that the lemma is true for all integers q less than the given one, noting that the conclusion is vacuously true if $q = 0$.

We have

$$H_{n,1}(\theta) = H'_{n,0}(\theta) + q\frac{\cos\theta H_{n,0}(\theta)}{\sin\theta} = H'_n(\theta) + q\cos\theta\frac{H_n(\theta)}{\sin\theta},$$

in which the magnitude of the second term may be estimated by Rolle's Theorem. We obtain, since $|(\theta\cos\theta)/\sin\theta| < 1$ for $|\theta| \le \pi/2$, with a similar estimate if $|\theta - \pi| \le \pi/2$,

$$\|H_{n,1}(\theta)\| \le (1+q)\|H'_n(\theta)\| \le C(q+1)\varepsilon_n/n^{q-1}.$$

Moreover,

$$\frac{H_{n,q}(\theta)}{\sin^{q-1}\theta} = \frac{H'_n(\theta)}{\sin^{q-1}\theta} + \frac{H_n(\theta)}{\sin^q\theta}\cdot q\cos\theta,$$

and therefore $H_{n,1}(\theta)/\sin^{q-1}\theta \to 0$ as $\theta \to 0, \pi$. And clearly $H_{n,1}(\theta)$ is $q-1$ times continuously differentiable.

The conclusion of the lemma will hold for $H_{n,1}$ provided that we show the existence of a K_1 such that for $k = 0, \dots, q-1$

$$\|H_{n,1}^{(k)}(\theta)\| \leq \frac{C \cdot K_1 \varepsilon_n}{n^{q-k-1}}.$$

To this end we note that

$$H_{n,1}^{(k)}(\theta) = H_{n,0}^{(k+1)}(\theta) + q \left(\frac{\cos\theta H_{n,0}(\theta)}{\sin\theta}\right)^{(k)},$$

and that only the magnitude of the second term requires further investigation. We have for arbitrary θ

$$\left|\left(\frac{\cos\theta H_{n,0}(\theta)}{\sin\theta}\right)^{(k)}\right| \leq \sum_{j=0}^{k} \binom{k}{j} |(\cos\theta)^{(k-j)}| \cdot \left|\left(\frac{H_n(\theta)}{\sin\theta}\right)^{(j)}\right|$$
$$\leq \sum_{j=0}^{k} \binom{k}{j} \left|\left(\frac{H_n(\theta)}{\sin\theta}\right)^{(j)}\right|.$$

In turn,

$$\left|\left(\frac{H_n(\theta)}{\sin\theta}\right)^{(j)}\right| \leq \sum_{l=0}^{j} \binom{j}{l} |(H_n^{(l)}(\theta)| \left|\left(\frac{1}{\sin\theta}\right)^{(j-l)}\right| \leq \sum_{l=0}^{j} \binom{j}{l} (j-l)! \left|\frac{H_n^{(l)}(\theta)}{\sin^{j-l+1}\theta}\right|$$
$$\leq \frac{1}{j+1} \sum_{l=0}^{j} \binom{j+1}{l} \left(\frac{\pi}{2}\right)^{j-l+1} \|H_n^{(j+1)}(\theta)\| \leq \frac{C \cdot C'}{n^{q-j-1}} \varepsilon_n,$$

for some constant C'.

Now we get

$$\|H_{n,1}^{(k)}(\theta)\| \leq \frac{C\varepsilon_n}{n^{q-k-1}} + q \sum_{j=0}^{k} \binom{k}{j} \frac{C \cdot C'}{n^{q-j-1}} \varepsilon_n \leq \frac{C \cdot K_1}{n^{q-k-1}} \varepsilon_n,$$

and the lemma holds with the function $H_{n,1}$ in place of the function H_n. Hence, there is a constant K_2 such that for $k = 2, \dots, q$ (shifting indices)

$$\|H_{n,k}\| \leq \frac{C \cdot K_2 \varepsilon_n}{n^{q-k}},$$

and we have already seen that

$$\|H_{n,1}\| \leq \frac{C(q+1)}{n^{q-1}} \varepsilon_n.$$

Therefore the lemma holds for H_n with constant $K = \max\{q+1, K_2\}$.

Lemma 5. *Let $G \in C^q[-1, 1]$ be such that $G^{(k)}(\pm 1) = 0$ for $k = 0, \dots, q$, and $H(\theta) := (G(\cos\theta))/\sin^q(\theta)$. Then for K_q depending only upon q*

$$E_{n-q}^*(D_\theta^q H) \leq K_q E_{n-q}(D_x^q G).$$

Proof. The lemma is obvious if $q = 0$. We assume that the lemma holds true for all values of q less than the given one. We note that

$$D_\theta H(\theta) = \frac{-1}{\sin^{q-1}\theta} \left(G'(\cos\theta) + q\cos\theta \frac{G(\cos\theta)}{1-\cos^2\theta}\right).$$

With $x = \cos\theta$, the expression in parentheses is a function of x that lies in $C^{q-1}[-1, 1]$ and satisfies

$$\left(G'(x) + \frac{qxG(x)}{1-x^2}\right)^{(k)} = 0, \qquad x = \pm 1,\ k = 0, \dots, q-1.$$

Therefore, in accordance with our assumption that the lemma operates for $q-1$, we have

$$\begin{aligned} E^*_{n-q}(D_\theta H) = E^*_{n-q}(D_\theta^{q-1}(D_\theta H)) &\le K_{q-1} E_{n-q}\left(D_x^{q-1}\left(G'(x) + \frac{qxG(x)}{1-x^2}\right)\right) \\ &\le K_{q-1}\left(E_{n-q}(D_x^q G(x)) + qE_{n-q}\left(D_x^{q-1}\left(\frac{xG(x)}{1-x^2}\right)\right)\right). \end{aligned}$$

Our proof is now completed by showing that

$$E_{n-q}\left(D_x^{q-1}\left(\frac{xG(x)}{1-x^2}\right)\right) \le (2^q - 1)E_{n-q}(D_x^q G(x)),$$

from which it would clearly follow that $K_q \le (2^q)K_{q-1}$. We have for $x \in [-1, 1]$ that

$$\begin{aligned} \left(\frac{xG(x)}{1-x^2}\right)^{(q-1)} &= \sum_{k=0}^{q-1}\binom{q-1}{k} G^{(q-1-k)}(x)\left(\frac{x}{1-x^2}\right)^{(k)} \\ &= \frac{1}{2}\sum_{k=0}^{q-1}\binom{q-1}{k} G^{(q-1-k)}(x)\left(\frac{1}{1-x}\right)^{(k)} \\ &\quad - \frac{1}{2}\sum_{k=0}^{q-1}\binom{q-1}{k} G^{(q-1-k)}(x)\left(\frac{1}{1+x}\right)^{(k)} \\ &= \frac{1}{2}\sum_{k=0}^{q-1}\binom{q-1}{k} k!\frac{G^{(q-1-k)}(x)}{(1-x)^{k+1}} \\ &\quad + \frac{1}{2}\sum_{k=0}^{q-1}(-1)^{k+1}\binom{q-1}{k} k!\frac{G^{(q-1-k)}(x)}{(1+x)^{k+1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} &E_{n-q}\left(\left(\frac{xG(x)}{1-x^2}\right)^{(q-1)}\right) \\ &\quad \le \frac{1}{2}\sum_{k=0}^{q-1}\binom{q-1}{k} k!\left[E_{n-q}\left(\frac{G^{(q-1-k)}(x)}{(1-x)^{k+1}}\right) + E_{n-q}\left(\frac{G^{(q-1-k)}(x)}{(1-x)^{k+1}}\right)\right]. \end{aligned}$$

We may examine for arbitrary k the expression

$$E_{n-q}\left(\frac{G^{(q-1-k)}(x)}{(1-x)^{k+1}}\right)$$

and estimate its magnitude. Let $P^*_{n-q}(x)$ be any polynomial of degree $n-q$ or less approximating the function $G^{(q-1-k)}(x)/(1-x)^{k+1}$. Using repeatedly

Cauchy's mean value theorem, we obtain

$$\begin{aligned}\frac{G^{(q-1-k)}(x)}{(1-x)^{k+1}} - P^*_{n-q}(x) &= \frac{G^{(q-k-1)}(x) - (1-x)^{k+1}P^*_{n-q}(x)}{(1-x)^{k+1}} \\ &= \frac{1}{(k+1)!}(G^{(q)}(y) - [(1-y)^{k+1}P^*_{n-q}(y)]^{(k+1)})\end{aligned}$$

for some point y between x and 1. Furthermore, if $Q_{n-q}(x)$ is a polynomial of best approximation of degree at most $n-q$ for $G^{(q)}(x)$, then we are free to choose $P^*_{n-q}(x)$ of degree $n-q$ or less such that

$$[(1-x)^{k+1}P^*_{n-q}(x)]^{(k+1)} = Q_{n-q}(x).$$

Using this particular choice for P^*_{n-q}, we obtain

$$E_{n-q}\left(\frac{G^{(q-k-1)}(x)}{(1-x)^{k+1}}\right) \le \left\|\frac{G^{(q-k-1)}(x)}{(1-x)^{k+1}} - P^*_{n-q}(x)\right\| \le \frac{1}{(k+1)!}E_{n-q}(G^{(q)}).$$

Similarly

$$E_{n-q}\left(\frac{G^{(q-k-1)}(x)}{(1-x)^{k+1}}\right) \le \frac{1}{(k+1)!}E_{n-q}(G^{(q)}).$$

It follows, inserting these inequalities into the previous estimate, that

$$E_{n-q}\left(\left(\frac{xG(x)}{1-x^2}\right)^{(q-1)}\right) \le \frac{1}{q}\sum_{k=0}^{q-1}\binom{q}{k+1}E_{n-q}(G^{(q)}).$$

This final inequality completes the proof of the lemma.

Proof of the theorem. Let $f \in C^q[-1,1]$. We first note that from f we can subtract a fixed polynomial $Q(x)$ of degree at most $2q+1$ that interpolates $f^{(0)}, \dots, f^{(q)}$ at ± 1, obtaining $f(x) - Q(x) = g(x)$, in which the function $g(x)$ satisfies $g^{(k)}(\pm 1) = 0$ for $k = 0, \dots, q$. It is clear that for $n > 2q$ we have $E_{n-k}(f^{(k)}) = E_{n-k}(g^{(k)})$ for $k = 0, \dots, q$. For, if $P_n^{(k)}$ of degree $\le n-k$ is a polynomial of best approximation for $f^{(k)}$, then

$$E_{n-k}(f^{(k)}) = \|f^{(k)} - P_n^{(k)}\| = \|g^{(k)} + Q^{(k)} - P_n^{(k)}\| \ge E_{n-k}(g^{(k)}).$$

And similarly, if a polynomial $P_n^{(k)}$ of degree $\le n-k$ is a polynomial of best approximation for $g^{(k)}$ then we have $E_{n-k}(g^{(k)}) = \|g^{(k)} - P_n^{(k)}\| = \|f^{(k)} - Q^{(k)} - P_n^{(k)}\| \ge E_{n-k}(f^{(k)})$. Therefore, our theorem need only be proven for the function g. Now, we define the function $h(\theta)$ by $h(\theta) := g(\cos\theta)/\sin^q\theta$. It follows immediately that $h(\theta)$ is a 2π-periodic function that is q times continuously differentiable, and furthermore, $h(\theta), \dots, h^{(q)}(\theta)$ are all zero at $\theta = 0$ and $\theta = \pi$. Moreover, $h(\theta)$ is even if q is even and odd if q is odd. According to Lemma 3, we choose a polynomial of approximation $T_{n-q}(\theta)$ of order $n-q$ for $h(\theta)$ and its derivatives. It is possible as a consequence to write $T_{n-q}(\theta) = \sin^q\theta Q_{n-2q}(\cos\theta)$, in which Q_{n-2q} is a polynomial of degree at most $n-2q$ and $Q_{n-2}(\cos\theta)$ is also zero at 0 and π. We then have

$$\begin{aligned}g(\cos\theta) - \sin^{2q}\theta Q_{n-2q}(\cos\theta) &= \sin^q\theta(h(\theta) - \sin^q\theta Q_{n-2q}(\cos\theta)) \\ &= \sin^q\theta(h(\theta) - T_{n-q}(\theta)).\end{aligned}$$

From this identity it follows immediately that

$$|g(x) - (1 - x^2)^q Q_{n-2q}(x)| \le \gamma_0(\sqrt{1 - x^2}/n)^q E^*_{n-q}(h).$$

Furthermore, defining $H_n(\theta) = h(\theta) - T_{n-q}(\theta)$, it is seen by Lemma 3 that

$$\|H_n^{(k)}(\theta)\| \le (\gamma_k/n^{q-k}) E^*_{n-q}(h^{(q)}).$$

Therefore, by Lemma 4, using $\varepsilon_n = E^*_{n-q}(h^{(q)})$ and $C = \max\{\gamma_0, \dots, \gamma_q\}$, it follows that

$$\|H_{n,k}(\theta)\| \le (CK/n^{q-k}) E^*_{n-q}(h^{(q)}),$$

and moreover we have for $k = 0, \dots, q-1$

$$\begin{aligned} \frac{1}{\sin\theta}(\sin^{q-k}\theta H_{n,k}(\theta))' &= \sin^{q-k-1}\theta \left(H'_{n,k}(\theta) + (q-k)\frac{\cos\theta H_{n,k}(\theta)}{\sin\theta}\right) \\ &= \sin^{q-k-1}\theta H_{n,k+1}(\theta). \end{aligned}$$

Consequently, we have for $k = 0, \dots, q$

$$|(g(x) - (1 - x^2)^q Q_{n-2q}(x))^{(k)}| \le C\dot{K}(\sqrt{1 - x^2}/n)^{q-k} E^*_{n-q}(h^{(q)})$$

and also $(g(x) - (1 - x^2)^q Q_{n-2q}(x))^{(q)} = 0$ at ± 1. Lemma 5 now shows that

$$E^*_{n-q}(h^{(q)}(\theta)) \le C \cdot K \cdot K_q E_{n-q}(g^{(q)}(x)).$$

This completes the proof of the theorem with $P_n(x) = (1 - x^2)^q Q_{n-2q}(x)$.

References

1. N. Achieser and M. Kreĭn, *On the best approximation of periodic functions*, Dokl. Akad. Nauk SSSR **15** (1937), 107–112. (Russian)
2. K. Balázs, T. Kilgore, and P. Vértesi, *An interpolatory version of Timan's theorem*, Acta Math. Hungar (to appear).
3. E. Cheney, *Introduction to approximation theory*, McGraw-Hill, New York, 1966.
4. J. Czipszer and G. Freud, *Sur l'approximation d'une fonction périodique et de ses dérivées successives*, Acta Math. **99** (1958), 33–51.
5. J. Favard, *Sur les meilleures procédés d'approximation de certaines classes des fonctions par des polynômes trigonométriques*, Bull. Sci. Math. (2) **61** (1937), 209–224, 243–256.
6. I. Gopengauz, *A theorem of A. F. Timan on the approximation of functions by polynomials on a finite segment*, Mat. Z. **1** (1967), 163–172 ; English transl., Math. Notes **1** (1967), 110–116.
7. T. Kilgore, *On simultaneous approximation*, Proceedings of the Conference on Approximation Theory, Kecskemet, Hungary, August 1990 (to appear).
8. D. Leviatan, *The behavior of the derivatives of the algebraic polynomials of best approximation*, J. Approx. Theory **35** (1982), 169–176.
9. G. Lorentz, *Approximation of functions*, Holt, Rinehart, and Winston, New York, 1966.
10. S. Telyakovskiĭ, *Two theorems on the approximation of functions by algebraic polynomials*, Mat. Sb. **70** (1966), 252–265.
11. A. Timan, *An extension of Jackson's Theorem on the best approximation of a continuous function*, Dokl. Akad. Nauk SSR **778** (1951), 17–20. (Russian)
12. R. Trigub, *Approximation of functions by polynomials with integral coefficients*, Izv. Akad. Nauk. SSSR Ser. Mat. **26** (1962), 261–280. (Russian)

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