PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 118, Number 2, June 1993

INEQUALITIES IN THE MOST SIMPLE SOBOLEV SPACE AND CONVOLUTIONS OF L_2 FUNCTIONS WITH WEIGHTS

SABUROU SAITOH

(Communicated by Andrew M. Bruckner)

ABSTRACT. For the most simple Sobolev space on $\mathbb{R}$ composed of real-valued and absolutely continuous functions $f(x)$ on $\mathbb{R}$ with finite norms

$$\left\{\int_{-\infty}^{\infty}(a^2f'(x)^2+b^2f(x)^2)\,dx\right\}^{1/2} \qquad (a, b>0),$$

we shall apply the theory of reproducing kernels, and derive natural norm inequalities in the space and the related inequalities for convolutions of L_2 functions with weights.

1. INTRODUCTION AND RESULTS

We shall examine the most simple Sobolev (Hilbert) space $H(a, b)$ on $\mathbb{R}$ composed of real-valued and absolutely continuous functions $f(x)$ with finite norms

$$\left\{\int_{-\infty}^{\infty}\{a^2f'(x)^2+b^2f(x)^2\}\,dx\right\}^{1/2} \qquad (a, b>0). \tag{1.1}$$

The space has been examined extensively by many authors and was first introduced by Hardy and Littlewood in 1932 in connection with the famous Hardy-Littlewood integral inequality. See, for example, Evans and Everitt [2] and its references for the details.

Note that the space $H(a, b)$ admits the reproducing kernel

$$G_{a,b}(x, y)=\frac{1}{2ab}e^{-\frac{b}{a}|x-y|} \tag{1.2}$$

satisfying the reproducing property

$$f(y)=(f(\cdot), G_{a,b}(\cdot, y))_{H(a,b)} \quad \text{for all } f\in H(a, b).$$

Therefore, from the point of view of the theory of reproducing kernels, we examine the space $H(a, b)$ and obtain the following norm inequality.

Received by the editors June 28, 1991 and, in revised form, October 8, 1991.
1991 *Mathematics Subject Classification.* Primary 26D10; Secondary 30C40.
Key words and phrases. Sobolev space, inequality, reproducing kernel, Green's function, convolution, norm inequality, absolutely continuous functions.

Theorem 1.1. *For any functions $f \in H(a_1, b_1)$ and $g \in H(a_2, b_2)$, we have the inequality*

$$(1.3)\quad \begin{aligned}\left[\frac{1}{2}\left(\frac{a_1}{b_1}+\frac{a_2}{b_2}\right)\right]^{-1} \int_{-\infty}^{\infty} &[(a_1a_2)^2\{(f(x)g(x))'\}^2 \\ &+ (a_1b_2+a_2b_1)^2(f(x)g(x))^2]\,dx \\ \le \int_{-\infty}^{\infty} (a_1^2 f'(x)^2 + b_1^2 f(x)^2)\,dx &\cdot \int_{-\infty}^{\infty} (a_2^2 g'(x)^2 + b_2^2 g(x)^2)\,dx .\end{aligned}$$

Equality holds here if and only if f and g are expressible in the forms

$$f(x) = c_1 G_{a_1, b_1}(x, y) \quad \text{and} \quad g(x) = c_2 G_{a_2, b_2}(x, y)$$

for some real constants c_1 and c_2, and for some point $y \in \mathbb{R}$. Here, if $c_1c_2 \neq 0$ and equality holds, then the point y must be a common point for f and g.

Theorem 1.1 can be transformed in the following form by means of Fourier's integral.

Theorem 1.2. *For any complex-valued functions $F \in L_2(\mathbb{R}, (a_1^2\xi^2 + b_1^2)^{-1}\,d\xi)$ and $G \in L_2(\mathbb{R}, (a_2^2\xi^2 + b_2^2)^{-1}\,d\xi)$, and for the convolution $*$, we have the inequality*

$$(1.4)\quad \begin{aligned}\left[\frac{1}{2}\left(\frac{a_1}{b_1}+\frac{a_2}{b_2}\right)\right]^{-1} \frac{1}{2\pi}\int_{-\infty}^{\infty} &\left|\left(\frac{F(\xi)}{a_1^2\xi^2+b_1^2} * \frac{G(\xi)}{a_2^2\xi^2+b_2^2}\right)(\xi)\right|^2 \\ &\cdot \{(a_1a_2)^2\xi^2 + (a_1b_2+a_2b_1)^2\}\,d\xi \\ \le \int_{-\infty}^{\infty} \frac{|F(\xi)|^2}{a_1^2\xi^2+b_1^2}\,d\xi &\int_{-\infty}^{\infty} \frac{|G(\xi)|^2}{a_2^2\xi^2+b_2^2}\,d\xi .\end{aligned}$$

Equality holds here if and only if F and G are expressible in the forms

$$F(\xi) = d_1 e^{i\xi y} \quad \text{and} \quad G(\xi) = d_2 e^{i\xi y}$$

for some complex numbers d_1 and d_2, and for some point $y \in \mathbb{R}$. Here, if $d_1 d_2 \neq 0$ and equality holds, then the point y must be a common point for F and G.

In connection with inequality (1.4), we derive inequalities of two types for $a_1 = a_2 = 0$ and $b_1 = b_2 = 1$, for functions $F \in L_2(0, \infty)$ and $G \in L_2(0, \infty)$ [4], and for functions $F \in L_2(\alpha, \beta)$ and $G \in L_2(\gamma, \delta)$ [5]. Their results seem to be quite different from (1.4).

As we will see from the proof of Theorem 1.1, our basic idea comes from the general theory of reproducing kernels by Aronszajn [1] in a general principle, but it will be difficult to derive a general version of Theorem 1.1. Indeed, we use a special property of $G_{a,b}(x, y)$. See (2.2) and (2.3).

2. Proof of the inequality in Theorem 1.1

To start with, note that $G_{a,b}(x, y)$ is the reproducing kernel for the space $H(a, b)$, and at the same time, it is the Green function for the differential equation satisfying

$$-a^2 D_x^2 G_{a,b}(x, y) + b^2 G_{a,b}(x, y) = \delta(x - y)$$

and

$$\lim_{x\to\infty} G_{a,b}(x, y) = \lim_{x\to-\infty} G_{a,b}(x, y) = 0$$

(cf. Hirschman and Widder [3, p. 9] and Shapiro [8, pp. 91–92]).

We recall the expression

$$G_{a,b}(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{a^2\xi^2 + b^2} e^{i\xi(x-y)}\, d\xi. \tag{2.1}$$

From the identity

$$G_{a_1,b_1}(x, y)G_{a_2,b_2}(x, y) = \frac{1}{4a_1a_2b_1b_2} e^{-(b_1/a_1+b_2/a_2)|x-y|}, \tag{2.2}$$

we obtain conversely (or directly) the identity

$$\begin{aligned} &G_{a_1,b_1}(x, y)G_{a_2,b_2}(x, y) \\ &\quad= \frac{1}{2}\left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}\, d\xi}{(a_1a_2)^2\xi^2 + (a_1b_2 + a_2b_1)^2}. \end{aligned} \tag{2.3}$$

Of course, $G_{a_1,b_1}(x, y)G_{a_2,b_2}(x, y)$ is a positive matrix (a positive definite) on $\mathbb{R}$, and so, there exists a uniquely determined functional Hilbert space H' admitting the reproducing kernel $G_{a_1,b_1}(x, y)G_{a_2,b_2}(x, y)$ [1, p. 344]. In our case, we see conversely that the space H' can be realized as the Sobolev space $H((a_1a_2), (a_1b_2 + a_2b_1))$ whose norm is given by

(2.4)

$$\left[\frac{1}{2}\left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right)\right]^{-1/2} \left[\int_{-\infty}^{\infty} \{(a_1a_2)^2 f'(x)^2 + (a_1b_2 + a_2b_1)^2 f(x)^2\}\, dx\right]^{1/2}.$$

Meanwhile, we consider the tensor product $H = H(a_1, b_1) \otimes H(a_2, b_2)$. Then, $G_{a_1,b_1}(x_1, y_1)G_{a_2,b_2}(x_2, y_2)$ is the reproducing kernel for the tensor product H [1, pp. 357–362]. Here, any members $h^{(j)}(x_1, x_2)$ in H are expressible in the form

$$h^{(j)}(x_1, x_2) = \sum_{n=1}^{\infty} f_n^{(j)}(x_1) g_n^{(j)}(x_2); \qquad f_n^{(j)} \in H(a_1, b_1),\ \ g_n^{(j)} \in H(a_2, b_2),$$

with finite norms

$$\|h^{(j)}\|_H^2 = \sum_{n=1}^{\infty} \|f_n^{(j)}\|_{H(a_1,b_1)}^2 \|g_n^{(j)}\|_{H(a_2,b_2)}^2.$$

Furthermore, the inner product $(h^{(1)}, h^{(2)})_H$ is given by

$$(h^{(1)}, h^{(2)})_H = \sum_{n=1}^{\infty} (f_n^{(1)}, f_n^{(2)})_{H(a_1,b_1)} (g_n^{(1)}, g_n^{(2)})_{H(a_2,b_2)}.$$

Next, we consider the Hilbert space H_r which is formed by restricting functions in H to the diagonal set of $\mathbb{R} \times \mathbb{R}$ formed by $\{(x, x);\ x \in \mathbb{R}\}$. We identify it with $\mathbb{R}$. For any such restriction $f \in H_r$, the norm in H_r is given by $\min\|h\|_H$ for all $h \in H$, the restriction of which to the diagonal set is f. Then, $G_{a_1,b_1}(x, y)G_{a_2,b_2}(x, y)$ is the reproducing kernel for the space H_r [1, p. 361, Theorem II]; that is, we have the identity $H_r = H'$ as a Hilbert space.

In particular, we have for any $f \in H(a_1, b_1)$ and $g \in H(a_2, b_2)$,

$$f(x)g(x) \in H_r,$$

and furthermore, we have the inequality

$$\|f(x)g(x)\|_{H_r}^2 \leq \|f(x_1)g(x_2)\|_H^2;$$

that is, inequality (1.3).

3. Proof of the equality statement in Theorem 1.1

We assume that for $f \in H(a_1, b_1)$ and $g \in H(a_2, b_2)$ $(f \not\equiv 0,\ g \not\equiv 0)$, equality holds in (1.3). Note that then $f(x_1)g(x_2)$ is characterized by the orthogonality

$$(f(x_1)g(x_2), h(x_1, x_2))_H = 0 \tag{3.1}$$

for any $h \in H$ satisfying

$$h(x, x) = 0 \quad \text{on } \mathbb{R}. \tag{3.2}$$

As functions satisfying (3.2), we take

$$\begin{aligned} h(x_1, x_2) &= G_{a_1, b_1}(x_1, y_1)G_{a_2, b_2}(x_2, y_2) \\ &\quad - \frac{G_{a_1, b_1}(x_1, y_1)G_{a_2, b_2}(x_1, y_2)G_{a_2, b_2}(x_2, 0)}{G_{a_2, b_2}(x_1, 0)} \\ &\in H \quad \text{for any } y_1, y_2 \in \mathbb{R}. \end{aligned}$$

Then, by using the reproducing property of $G_{a_1, b_1}(x_1, y_1)G_{a_2, b_2}(x_2, y_2)$ in H, we have the following from (3.1):

$$\begin{aligned} f(y_1)g(y_2) &= g(0)\left(f(x_1), \frac{G_{a_1, b_1}(x_1, y_1)G_{a_2, b_2}(x_1, y_2)}{G_{a_2, b_2}(x_1, 0)}\right)_{H(a_1, b_1)} \\ &= g(0)\int_{-\infty}^{\infty} G_{a_1, b_1}(x_1, y_1) \\ &\qquad \cdot \{-a_1^2 f''(x_1) + b_1^2 f(x_1)\}\frac{G_{a_2, b_2}(x_1, y_2)}{G_{a_2, b_2}(x_1, 0)}\, dx_1. \end{aligned} \tag{3.3}$$

By using the property of $G_{a_1, b_1}(x_1, y_1)$ as Green's function, we have the identity

$$\{-a_1^2 f''(y_1) + b_1^2 f(y_1)\}g(y_2) = g(0)\{-a_1^2 f''(y_1) + b_1^2 f(y_1)\}\frac{G_{a_2, b_2}(y_1, y_2)}{G_{a_2, b_2}(y_1, 0)}. \tag{3.4}$$

Note that $-a_1^2 f''(y_1) + b_1^2 f(y_1) \not\equiv 0$ on $\mathbb{R}$.

Indeed, if $-a_1^2 f''(x) + b_1^2 f(x) \equiv 0$ on $\mathbb{R}$, then we have, for some constants C_1 and C_2,

$$f(x) = C_1 e^{-b_1 x/a_1} + C_2 e^{b_1 x/a_1} \quad \text{on } \mathbb{R}.$$

Hence, from the fact that $f \in H(a_1, b_1)$ we see that $C_1 = C_2 = 0$.

Therefore, from (3.4) we have, for at least one point y_0,

$$g(y_2) = g(0)\frac{G_{a_2, b_2}(y_0, y_2)}{G_{a_2, b_2}(y_0, 0)} = g(0)\frac{G_{a_2, b_2}(y_2, y_0)}{G_{a_2, b_2}(y_0, 0)}. \tag{3.5}$$

We thus have the expression of y

$$g(x) = c_1 G_{a_2, b_2}(x, \alpha)$$

for some real constant c_1 and for some point $\alpha \in \mathbb{R}$. Similarly, we have the expression of f in the form

$$f(x) = c_2 G_{a_1, b_1}(x, \beta) \tag{3.6}$$

for some real constant c_2 and for some point $\beta \in \mathbb{R}$.

From (3.3) and (3.6) we have

$$\begin{aligned} f(y_1)g(y_2) &= g(0)c_2 \frac{G_{a_1, b_1}(\beta, y_1)G_{a_2, b_2}(\beta, y_2)}{G_{a_2, b_2}(\beta, 0)} \\ &= c_2 g(0) \frac{G_{a_1, b_1}(y_1, \beta)G_{a_2, b_2}(y_2, \beta)}{G_{a_2, b_2}(\beta, 0)}. \end{aligned}$$

Conversely, for this function, the condition (3.1) for functions $h \in H$ satisfying (3.2) is apparently satisfied, as we see from the reproducing property of $G_{a_1, b_1}(x_1, \beta)G_{a_2, b_2}(x_2, \beta)$ in H. We thus have the desired result.

4. Proof of Theorem 1.2

First note that from the representation (2.1) of $G_{a_1, b_1}(x, y)$, any member $f \in H(a_1, b_1)$ is expressible in the form

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(\xi)}{a_1^2 \xi^2 + b_1^2} e^{i\xi x}\, d\xi \tag{4.1}$$

for a complex-valued function F satisfying

$$\int_{-\infty}^{\infty} \frac{|F(\xi)|^2}{a_1^2 \xi^2 + b_1^2}\, d\xi < \infty, \tag{4.2}$$

and furthermore we have the isometrical identity

$$\|f\|^2_{H(a_1, b_1)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|F(\xi)|^2}{a_1^2 \xi^2 + b_1^2}\, d\xi \tag{4.3}$$

[6, Theorem 3.1; 7, p. 82, Theorem 3.1]. Here, note that the Sobolev space $H(a_1, b_1)$ admitting the reproducing kernel $G_{a_1, b_1}(x, y)$ can be considered for complex-valued functions $f(x)$ satisfying (4.1) and (4.2). In order to consider the general complex-valued L_2 functions satisfying (4.2), we shall consider the Sobolev spaces $H(a_1, b_1)$ and $H(a_2, b_2)$ as complex Hilbert spaces in the sequel.

Similarly, we have the expressions

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(\xi)}{a_2^2 \xi^2 + b_2^2} e^{i\xi x}\, d\xi \tag{4.4}$$

and

$$\|g\|^2_{H(a_2, b_2)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|G(\xi)|^2}{a_2^2 \xi^2 + b_2^2}\, d\xi. \tag{4.5}$$

From (4.1) and (4.4), we have the expression

$$f(x)g(x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left(\frac{F(\xi)}{a_1^2 \xi^2 + b_1^2} * \frac{G(\xi)}{a_2^2 \xi^2 + b_2^2} \right)(\xi) e^{i\xi x}\, d\xi. \tag{4.6}$$

By using expression (2.3), we have similarly the identity

(4.7)
$$\|fg\|_{H_r}^2 = \left[\frac{1}{2}\left(\frac{a_1}{b_1}+\frac{a_2}{b_2}\right)\right]^{-1}\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\left|\left(\frac{F(\xi)}{a_1^2\xi^2+b_1^2} * \frac{G(\xi)}{a_2^2\xi^2+b_2^2}\right)(\xi)\right|^2 \cdot \{(a_1a_2)^2\xi^2+(a_1b_2+a_2b_1)^2\}\,d\xi.$$

We thus obtain Theorem 1.2 from Theorem 1.1 for complex-valued functions, directly.

ACKNOWLEDGMENTS

The author would like to thank the referee for his careful reading and valuable comments.

REFERENCES

1. N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.
2. W. D. Evans and W. N. Everitt, *A return to the Hardy-Littlewood integral inequality*, Proc. Roy. Soc. London Ser. A **380** (1982), 447–486.
3. I. I. Hirschman and D. V. Widder, *The convolution transform*, Princeton Univ. Press, Princeton, NJ, 1955.
4. S. Saitoh, *Hilbert spaces induced by Hilbert space valued functions*, Proc. Amer. Math. Soc. **89** (1983), 74–78.
5. ——, *A fundamental inequality in the convolution of L_2 functions on the half line*, Proc. Amer. Math. Soc. **91** (1984), 285–286.
6. ——, *On the convolution of L_2 functions*, Kodai Math. J. **9** (1986), 50–57.
7. ——, *Theory of reproducing kernels and its applications*, Pitman Res. Notes Math. Ser., vol. 189, Longman Sci. Tech., Harlow, England, 1988.
8. H. S. Shapiro, *Topics in approximation theory*, Lecture Notes in Math., vol. 187, Springer-Verlag, Berlin and New York, 1971.

DEPARTMENT OF MATHEMATICS, FACULTY OF ENGINEERING, GUNMA UNIVERSITY, KIRYU 376, JAPAN