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EXTENSION OF HOLOMORPHIC MAPPINGS FROM E TO E''

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ABSTRACT. Assuming that E is a distinguished locally convex space and F is a complete locally convex space, we prove that there exists an open subset V of E'' that contains E and such that every holomorphic mapping $f: E \to F$ whose restriction $f|B$ is $\sigma(E, E')$-uniformly continuous for every bounded subset B of E has a unique holomorphic extension $\tilde{f}: V \to F$ such that $\tilde{f}|B$ is $\sigma(E'', E')$-uniformly continuous for every bounded subset B of V. We show that in many cases we can take $V = E''$. This is the case when E'' is a locally convex space where every G-holomorphic mapping that is bounded in a neighbourhood of the origin is locally bounded.

INTRODUCTION

Given locally convex spaces E and F, we consider the problem of extending an analytic mapping $f: E \to F$ to an analytic mapping $\tilde{f}: E'' \to F$. It is clear that if we have such extension of f to E'', then we can extend this f to every locally convex space G such that $E \subset G$ and there exists $S: G \to E''$ linear, continuous with $S|E = \mathrm{id}_E$. In case of Banach spaces we know that E is an $\mathscr{L}_\infty$-space in the sense of Lindenstrauss and Pełczynski if and only if for every locally convex space G that contains E as a subspace there exists $S: G \to E''$ linear, continuous and such that $S|E = \mathrm{id}_E$ (cf. [10, Example 2(c)]). The spaces c_o, l_∞, $L_\infty(\mu)$, and $C(K)$ are examples of such spaces.

We recall that the problem of extending an analytic mapping was asked by Dineen in [3]. The first general positive answer to Dineen's question was given by Boland in [2]; namely, he proved that if F is a closed subspace of a dual G of a nuclear Fréchet space then every holomorphic function on F has an extension to a holomorphic function on G. Since then, a lot of progress has been made, mainly for holomorphic functions on (DFN)-spaces. Meise and Vogt gave in [7] an example of a Fréchet nuclear space G where the holomorphic Hahn-Banach theorem is not valid. The case of Banach spaces was studied first by Aron and Berner in [1]. They showed that every holomorphic function on a Banach space E that is bounded on the bounded subsets of E can be extended to a holomorphic function on E'' that is bounded on the bounded subsets of E''. As a consequence they proved that a holomorphic function defined on c_o can

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be extended to a holomorphic function on l_∞ if and only if it is bounded on every bounded subset of c_o. We are going to consider classes of holomorphic mappings defined on locally convex spaces. This paper generalizes results of [8, 9].

Notation and terminology

Let E and F be complex Hausdorff locally convex spaces. Given a subset A of E, we denote by A^o the polar of A with respect to $\sigma(E', E)$ and by A^{oo} the polar of A^o with respect to $\sigma(E'', E')$. The set of all continuous seminorms on F is indicated by $\mathrm{CS}(F)$ and the set of all neighbourhoods of $x \in E$ is indicated by $\mathscr{U}_E(x)$; given a subset X of E, the set of all bounded subsets B of E such that $B \subset X$ is denoted by $\mathscr{B}(X)$ and the set of all absolutely convex elements of $\mathscr{B}(X)$ is denoted by $\mathscr{B}_{\mathrm{ac}}(X)$. The topology on E' of uniform convergence on the bounded subsets of E is denoted by β; E'_β and $(E'_\beta)'_\beta$ are written E' and E'', respectively. If X is a subset of E and $\alpha \in \mathrm{CS}(F)$, we define $\|f\|_{\alpha, X} := \sup\{\alpha \circ f(x) : x \in X\}$ for every mapping $f: E \to F$.

As usual, $H_G(E, F)$ denotes the space of all G-holomorphic mappings from E into F, $H(E, F)$ denotes the space of all holomorphic mappings from E into F, and $\mathscr{P}(^nE, F)$ denotes the space of all continuous n-homogeneous polynomials from E into F. We recall that $P: E \to F$ is an n-homogeneous polynomial if and only if there exists an n-linear mapping $A: E^n \to F$ such that $P(x) = A(x, \dots, x)$ for all $x \in E$; in this case we denote $P = \widehat{A}$.

For all $n \in N$, let $\mathscr{L}_{a,wu}(^nE, F)$ be the space of all n-linear mappings $A: E^n \to F$ such that, for every $B \in \mathscr{B}(E)$, $A|B^n$ is uniformly continuous on $(B, \sigma(E, E'))^n$. The space of all elements of $\mathscr{L}_{a,wu}(^nE, F)$ that are continuous is denoted by $\mathscr{L}_{wu}(^nE, F)$.

By definition $H_b(E, F) := \{f \in H(E, F): \|f\|_{\alpha, B} < \infty \ \ \forall \alpha \in \mathrm{CS}(F), \ \forall B \in \mathscr{B}(E)\}$ and τ_b is the locally convex topology on $H_b(E, F)$ generated by the seminorms $\| \ \|_{\alpha, B}$ when B ranges over $\mathscr{B}(E)$ and α ranges over $\mathrm{CS}(F)$.

We will be particularly interested in the following spaces:

$$\mathscr{P}_{wu}(^nE, F) := \{P \in \mathscr{P}(^nE, F): P|B \text{ is uniformly } \sigma(E, E')\text{-continuous } \forall B \in \mathscr{B}(E)\},$$

$$\mathscr{P}_{w^*u}(^nE'', F) := \{P \in \mathscr{P}(^nE'', F): P|B^{oo} \text{ is (uniformly) } \sigma(E'', E')\text{-continuous } \forall B \in \mathscr{B}(E)\},$$

$$H_G^{wu}(E, F) := \{f \in H_G(E, F): f|B \text{ is uniformly } \sigma(E, E')\text{-continuous } \forall B \in \mathscr{B}(E)\},$$

$$H_G^{w^*u}(E'', F) := \{f \in H_G(E'', F): f|B^{oo} \text{ is (uniformly) } \sigma(E'', E')\text{-continuous } \forall B \in \mathscr{B}(E)\}.$$

Let $H^{wu}(E, F) := H_G^{wu}(E, F) \cap H(E, F)$ and $H^{w^*u}(E'', F) := H_G^{w^*u}(E'', F) \cap H(E'', F)$. We remark that if $f \in H^{wu}(E, F)$ then $d^nf(x) \in \mathscr{P}_{wu}(^nE, F)$ for all $x \in E$ and for all $n \in \mathbf{N}$; if $f \in H^{w^*u}(E'', F)$ then $d^nf(x) \in \mathscr{P}_{w^*u}(^nE'', F)$ for all $x \in E''$ and for all $n \in \mathbf{N}$.

A locally convex space E is said to be *distinguished* if every $\sigma(E'', E')$-bounded subset of its bidual E'' is contained in the $\sigma(E'', E')$-closure of some $B \in \mathscr{B}(E)$. For further notation and basic results we refer to [4–6].

The extension theorem

We are grateful to the referee who improved the original version of this paper by proving the following result:

Lemma 1. *Let E be a locally convex space. Then for every neighbourhood V of zero in E the set $V^{\times\times} := \bigcup_{B\in\mathscr{B}_{\mathrm{ac}}(V)} B^{oo}$ is a neighbourhood of zero in E''.*

Proof. Without loss of generality we may suppose V closed and absolutely convex. Since $E'' = \bigcup_{B\in\mathscr{B}_{\mathrm{ac}}(E)} B^{oo}$, it is clear that

$$V^{oo} = \bigcup_{B\in\mathscr{B}_{\mathrm{ac}}(E)} B^{oo}\cap V^{oo}. \tag{1}$$

We claim that

$$(V\cap B)^{oo} \supset \frac{1}{2}(V^{oo}\cap B^{oo}) \tag{2}$$

for all $B\in\mathscr{B}_{\mathrm{ac}}(E)$. If (2) is true,

$$\begin{aligned} V^{\times\times} &= \bigcup_{B\in\mathscr{B}_{\mathrm{ac}}(V)} B^{oo} = \bigcup_{B\in\mathscr{B}_{\mathrm{ac}}(E)} (B\cap V)^{oo} \supset \frac{1}{2}\bigcup_{B\in\mathscr{B}_{\mathrm{ac}}(E)} (V^{oo}\cap B^{oo}) \\ &= \frac{1}{2}\left(V^{oo}\cap \bigcup_{B\in\mathscr{B}_{\mathrm{ac}}(E)} B^{oo}\right) = \frac{1}{2}V^{oo} \end{aligned}$$

and since $V\in\mathscr{U}_E(0)$, V^o is an equicontinuous subset of E' (and hence a bounded subset of E') and so V^{oo} is a neighbourhood of zero in E''. Thus it suffices to show (2). Since V is absolutely convex and closed, it is $\sigma(E, E')$-closed and we have

$$(V\cap B)^{oo} = (\overline{\Gamma(V^o\cup B^o)}^{\sigma(E',E)})^o$$

where $\Gamma(V^o\cup B^o)$ is the convex hull of $V^o\cup B^o$. As V^o and B^o are absolutely convex sets, it is easy to verify that

$$\Gamma(V^o\cup B^o)\subset V^o+B^o\subset 2\Gamma(V^o\cup B^o). \tag{3}$$

Since V^o is an equicontinuous $\sigma(E', E)$-closed set, by Alaoglu-Bourbaki we have that V^o is $\sigma(E', E)$-compact. Now V^o is $\sigma(E', E)$-compact and B^o is $\sigma(E', E)$-closed, and thus V^o+B^o is $\sigma(E', E)$-closed. Hence

$$\overline{\Gamma(V^o\cup B^o)}^{\sigma(E',E)}\subset V^o+B^o$$

and $(V^o+B^o)^o\subset(\overline{\Gamma(V^o\cup B^o)}^{\sigma(E',E)})^o = (V\cap B)^{oo}$; using the second part of (3) we have $V^o+B^o\subset 2\Gamma(V^o\cup B^o)$ and so $(V^o+B^o)^o\supset(2\Gamma(V^o\cup B^o))^o = \frac{1}{2}(V^o\cup B^o)^o$ since $(\Gamma(A))^o = A^o$. Finally $(V^o\cup B^o)^o = V^{oo}\cap B^{oo}$, and so $\frac{1}{2}(V^{oo}\cap B^{oo}) = \frac{1}{2}(V^o\cup B^o)^o\subset(V^o+B^o)^o\subset(V\cap B)^{oo}$ and (2) is true.

Proposition 2. *Let E and F be locally convex spaces, F complete. Then for every $A\in\mathscr{L}_{a,wu}({}^nE, F)$ there is a unique extension $\widetilde{A}\in\mathscr{L}_a({}^nE'', F)$ such that for every $B\in\mathscr{B}(E)$ the restriction of $\widetilde{A}$ to $(B^{oo}, \sigma(E'', E'))^n$ is uniformly* *continuous. Moreover, for every* $\alpha \in \mathrm{CS}(F)$ *we have* $\|A\|_{\alpha, B^n} = \|\widetilde{A}\|_{\alpha,(B^{oo})^n}$ *for all* $B \in \mathscr{B}(E)$ *and the mapping* $T_n: \mathscr{L}_{a,wu}({}^nE, F) \to \mathscr{L}_a({}^nE'', F)$ *defined by* $T_n(A) := \widetilde{A}$ *for every* $A \in \mathscr{L}_{a,wu}({}^nE, F)$ *is linear and injective. If* $A \in \mathscr{L}_{a,wu}({}^nE, F)$ *is symmetric then* $\widetilde{A}$ *is symmetric as well.*

Proof. Take $A \in \mathscr{L}_{a,wu}({}^nE, F)$. Since for a given $B \in \mathscr{B}_{\mathrm{ac}}(E)$ the set B^n is dense in $(B^{oo}, \sigma(E'', E'))^n$, $A|B^n$ is uniformly continuous on $(B, \sigma(E, E'))^n$ and $\sigma(E'', E')|E = \sigma(E, E')$, by [6, Theorem 2, p. 61] there is a unique uniformly continuous mapping $\widetilde{A}^B: (B^{oo})^n \to F$ that extends $A|B^n$. By the uniqueness of extensions $\widetilde{A}^C|(B^{oo})^n = \widetilde{A}^B$ whenever $B \subset C$, this shows that $\widetilde{A}(x) := \widetilde{A}^B(x)$ if $x \in B^{oo}$ defines an n-linear mapping from $(E'')^n = (\bigcup_{B \in \mathscr{B}_{\mathrm{ac}}(E)} B^{oo})^n$ into F. The other statements are obvious by the density and uniqueness of the extension.

Proposition 3. *Let* E *be a locally convex space and let* F *be a complete locally convex space. Then for every* $m \in \mathbf{N}$ *there is a unique isomorphism (onto)*

$$\widehat{T}_m: \mathscr{P}_{wu}({}^mE, F) \to \mathscr{P}_{w^*u}({}^mE'', F)$$

such that

(1) $\widehat{T}_m P|E = P$ *for all* $P \in \mathscr{P}_{wu}({}^mE, F)$.

(2) *For every* $\alpha \in \mathrm{CS}(F)$, $\|\widehat{T}_m P\|_{\alpha, B^{oo}} = \|P\|_{\alpha, B}$ *for all* $B \in \mathscr{B}_{\mathrm{ac}}(E)$.

Proof. Let T_m be as in Proposition 2 and define $\widehat{T}_m\widehat{A} := (T_mA)^\wedge$ for every $A \in \mathscr{L}_{wu}({}^mE, F)$, i.e., $\widehat{T}_m\widehat{A}(x) := T_mA(x, \dots, x)$ for all $x \in E''$. Since (1) and (2) follow directly from Proposition 2, all we have to show is that $\widehat{T}_m\widehat{A}$ is continuous whenever $A \in \mathscr{L}_{wu}({}^mE, F)$. Since $\widehat{A}$ is continuous, there exists $V \in \mathscr{U}_E(0)$ such that V is absolutely convex and $\|\widehat{A}\|_{\alpha, V} < \infty$. For each $B \in \mathscr{B}_{ac}(V)$ it is clear that

$$\|\widehat{T}_m\widehat{A}\|_{\alpha, B^{oo}} \le \|T_mA\|_{\alpha,(B^{oo})^m} = \|A\|_{\alpha, B^m} \le \frac{m^m}{m!}\|\widehat{A}\|_{\alpha, B} \le \frac{m^m}{m!}\|\widehat{A}\|_V.$$

So, $\|\widehat{T}_m\widehat{A}\|_{\alpha, V^{\times\times}} \le (m^m/m!)\|\widehat{A}\|_V < \infty$ and so $\widehat{T}_m\widehat{A}$ is continuous by [4, Proposition 1.14 and Corollary 1.15].

Lemma 4. *Let* E *and* F *be locally convex spaces and let* $f: E \to F$ *be a mapping that is weakly uniformly continuous on each bounded subset of* E. *Then* $f(B)$ *is precompact for every* $B \in \mathscr{B}(E)$.

Proof. Since f is weakly uniformly continuous on B, given $V \in \mathscr{U}_F(0)$ there exist $\varphi_1, \dots, \varphi_k \in E'$ such that whenever $x, y \in B$ with $|\varphi_i(x-y)| < 1$ for all $i = 1, \dots, k$, $f(y) \in f(x) + V$. As the mapping

$$\begin{aligned} \psi: E &\to \mathbf{C}^k \\ x &\mapsto (\varphi_1(x), \dots, \varphi_k(x)) \end{aligned}$$

is continuous, we have $\psi(B)$ precompact in $\mathbf{C}^k$ (which we endow with the sup norm). So there exists $x_1, \dots, x_n \in B$ such that given any $x \in B$ there exists x_j $(1 \le j \le n)$ such that $|\varphi_i(x) - \varphi_i(x_j)| < 1$ for every $i = 1, \dots, k$. Thus given $x \in B$ there exists x_j $(1 \le j \le n)$ such that $f(x) \in f(x_j) + V$, which shows that $f(B) \subset \bigcup_{j=1}^n f(x_j) + V$ $(f(x_j) \in f(B))$. So $f(B)$ is precompact.

Corollary 5. *Let E and F be locally convex spaces. Then* $H^{wu}(E, F) \subset H_b(E, F)$.

Lemma 6. *Let E, F be locally convex spaces and $f \in H_G(E'', F)$ such that $\|f\|_{\alpha, B^{oo}} < \infty$ for all $B \in \mathscr{B}(E)$ and for all $\alpha \in \mathrm{CS}(F)$. If for all $y \in E''$, $f(y) = \sum_{k=0}^{\infty} P_k(y)$ with $P_k \in \mathscr{P}_{w^*u}({}^kE'', F)$ for all $k \in \mathbf{N}$, then $f \in H_G^{w^*u}(E'', F)$.*

Proof. Let $B \in \mathscr{B}(E)$ and $\alpha \in \mathrm{CS}(F)$. Using the Cauchy inequalities we get

$$\left\| f - \sum_{k=0}^{n} P_k \right\|_{\alpha, B^{oo}} \leq \left(\sum_{k=n+1}^{\infty} \frac{1}{2^k} \right) \cdot \|f\|_{\alpha, 2B^{oo}} \to 0 \quad \text{as } n \to \infty$$

since $\|f\|_{\alpha, 2B^{oo}} < \infty$ by hypothesis. Since $(\sum_{k=0}^{n} P_k)|B^{oo}$ is uniformly $\sigma(E'', E')$-continuous for every n and B^{oo} is $\sigma(E'', E')$-compact, $f|B^{oo}$ is uniformly $\sigma(E'', E')$-continuous.

Remark 7. If E is distinguished then $H_G^{w^*u}(E'', F) = \{f \in H_G(E'', F) : f|X$ is $\sigma(E'', E')$-continuous for every $X \in \mathscr{B}(E'')\}$, which is trivially contained in $H_G^{wu}(E'', F)$. Let τ_b be the locally convex topologies on $H_G^{wu}(E, F)$ and $H_G^{w^*u}(E'', F)$ generated by the seminorms $\|f\|_{\alpha, B} = \sup\{\alpha \circ f(x) : x \in B\}$ when α ranges over $\mathrm{CS}(F)$ and B ranges over $\mathscr{B}(E)$ and $\mathscr{B}(E'')$, respectively.

Theorem 8. *Let E be a locally convex space and let F be a complete locally convex space. Then for each $f \in H^{wu}(E, F)$ there exists a unique $\tilde{f} \in H_G^{w^*u}(E'', F)$ such that $\tilde{f}|E = f$. Moreover for each $\alpha \in \mathrm{CS}(F)$ there exists $W \in \mathscr{U}_{E''}(0)$ such that $\|\tilde{f}\|_{\alpha, W} < \infty$. If in addition E is distinguished, the mapping $Tf := \tilde{f}$ is a continuous linear mapping from $(H^{wu}(E, F), \tau_b)$ into $(H_G^{w^*u}(E'', F), \tau_b)$.*

Proof. Unicity follows from density of B in $(B^{oo}, \sigma(E'', E'))^n$. Given $f \in H^{wu}(E, F)$, let us define

$$\tilde{f}(y) := \sum_{k=0}^{\infty} \widehat{T}_k \left(\frac{\hat{d}^k f(0)}{k!} \right)(y) \quad \text{for all } y \in E'',$$

where $\widehat{T}_k$ is the unique isomorphism defined in Proposition 3. First of all we want to show that $\tilde{f}(y) \in F$ for all $y \in E''$. Let $P_k = \hat{d}^k f(0)/k!$ and $\widetilde{P}_k = \widehat{T}_k P_k$ for all $k = 0, 1, 2, \ldots$. Given any $y \in E''$ there exists a $B \in \mathscr{B}_{\mathrm{ac}}(E)$ such that $y \in B^{oo}$. For each $\alpha \in \mathrm{CS}(F)$,

$$\alpha \left(\sum_{k=0}^{N} \widetilde{P}_k(y) \right) \leq \sum_{k=0}^{N} \alpha \circ \widetilde{P}_k(y) \quad \text{for all } N = 0, 1, 2, \ldots.$$

So if $\sum_{k=0}^{N} \alpha \circ \widetilde{P}_k(y)$ converges when $N \to \infty$ for every $\alpha \in \mathrm{CS}(F)$, then $(\sum_{k=0}^{N} \widetilde{P}_k(y))_{N=0}^{\infty}$ is a Cauchy sequence in F that is complete, and we infer that $\sum_{k=0}^{N} \widetilde{P}_k(y)$ converges in F as $N \to \infty$. Now, for each $\alpha \in \mathrm{CS}(F)$ we have

$$\begin{aligned} \sum_{k=0}^{N} \alpha \circ \widetilde{P}_k(y) &\leq \sum_{k=0}^{N} \|\widetilde{P}_k\|_{\alpha, B^{oo}} = \sum_{k=0}^{N} \|P_k\|_{\alpha, B} \\ &\leq \left(\sum_{k=0}^{N} \frac{1}{2^k} \right) \cdot \|f\|_{\alpha, 2B} \to 2 \cdot \|f\|_{\alpha, 2B} < \infty \quad \text{as } N \to \infty. \end{aligned}$$

Consequently $\tilde{f}(y) := \sum_{k=0}^{\infty} \widetilde{P}_k(y) \in F$ for every $y \in E''$ and so $\tilde{f} \in H_G(E'', F)$. Let us show that $\tilde{f} \in H_G^{w^*u}(E'', F)$; it follows from Lemma 6 and

$$\|\tilde{f}\|_{\alpha, B^{oo}} \leq \sum_{k=0}^{\infty} \|\widetilde{P}_k\|_{\alpha, B^{oo}} = \sum_{k=0}^{\infty} \|P_k\|_{\alpha, B} \leq \left(\sum_{k=0}^{\infty} \frac{1}{2^k}\right) \|f\|_{\alpha, 2B} < \infty$$

for all $\alpha \in \mathrm{CS}(F)$ and for every $B \in \mathscr{B}(E)$. The τ_b-continuity of T, when E is distinguished, follows from $\|Tf\|_{\alpha, B^{oo}} \leq 2\|f\|_{\alpha, 2B}$ for every $\alpha \in \mathrm{CS}(F)$ and $B \in \mathscr{B}(E)$.

Finally, since $f \in H(E, F)$, there exists $V \in \mathscr{U}_E(0)$ absolutely convex such that $\|f\|_{\alpha, V} \leq M_\alpha < \infty$ for each $\alpha \in \mathrm{CS}(F)$. So, for every $k = 0, 1, 2, \ldots$,

$$\begin{aligned}\|\widetilde{P}_k\|_{\alpha, (V/2)^{\times\times}} &= \sup_{L \in \mathscr{B}_{\mathrm{ac}}(V/2)} \|\widetilde{P}_k\|_{\alpha, L^{oo}} = \sup_{L \in \mathscr{B}_{\mathrm{ac}}(V/2)} \|P_k\|_{\alpha, L} \\ &\leq \frac{1}{2^k}\|f\|_{\alpha, 2L} \leq \frac{1}{2^k} M_\alpha\end{aligned}$$

and consequently $\|\tilde{f}\|_{\alpha, (V/2)^{\times\times}} < \infty$. Now it is enough to remember that $(\frac{1}{2}V)^{\times\times} \in \mathscr{U}_{E''}(0)$ by Lemma 1.

Remark 9. (1) If E is a locally convex space such that every $f \in H_G(E'', F)$ that is bounded in a neighbourhood of the origin is locally bounded, then we may write $H^{w^*u}(E'', F)$ instead of $H_G^{w^*u}(E'', F)$ in Theorem 8 and T is an isomorphism between $(H^{wu}(E, F), \tau_b)$ and $(H^{w^*u}(E'', F), \tau_b)$ if E is distinguished. For instance, all (DFC)-spaces E satisfy the above conditions.

(2) If E is a bornological space that contains a fundamental sequence of bounded sets $(B_n)_{n=1}^{\infty}$ and such that E' is distinguished, then by [10, Proposition 8] of $\tilde{f}$ is locally bounded and so $\tilde{f} \in H^{w^*u}(E'', F)$ and T is a topological isomorphism between $(H^{wu}(E, F), \tau_b)$ and $(H^{w^*u}(E'', F), \tau_b)$.

Theorem 10. *Let E be a distinguished locally convex space and let F be a complete locally convex space. Then for each $f \in H^{wu}(E, F)$ there exists an open subset U of E'' and a unique $\tilde{f} \in H(U, F)$ such that*

(1) $E \subset U$,

(2) $\tilde{f}|E = f$.

If in addition E is distinguished, we have

(3) $\tilde{f}|X$ *is $\sigma(E'', E')$-continuous for every $X \in \mathscr{B}(U)$.*

Proof. From Theorem 8 there exists a unique $\tilde{f} \in H_G^{w^*u}(E'', F)$ such that $\tilde{f}|E = f$. For each $a \in E$ we define $f_a : E \to F$ by $f_a(x) := f(x+a)$. It is clear that $f_a \in H^{wu}(E, F)$. Now we prove as in Theorem 8 that there exists a unique $\tilde{f}_a \in H_G^{w^*u}(E'', F)$ such that $\tilde{f}_a|E = f_a$ and there exists $V_a \in \mathscr{U}_E(0)$ absolutely convex such that $\|\tilde{f}_a\|_{\alpha, (V_a/2)^{\times\times}} < \infty$ for every $\alpha \in \mathrm{CS}(F)$. Let $\sigma_a(x) := x + a$ for every $x \in E''$. It is clear that $f_a = f \circ \sigma_a|E$ and $\tilde{f} \circ \sigma_a|E = f_a = \tilde{f}_a|E$. Since $\tilde{f} \circ \sigma_a \in H_G^{w^*u}(E'', F)$, the uniqueness of the extension gives $\tilde{f}_a = \tilde{f} \circ \sigma_a$. So

$$\|\tilde{f}\|_{\alpha, a+(V_a/2)^{\times\times}} = \|\tilde{f} \circ \sigma_a\|_{\alpha, (V_a/2)^{\times\times}} = \|\tilde{f}_a\|_{\alpha, (V_a/2)^{\times\times}} < \infty.$$

If W_a is the interior of $(\frac{1}{2}V_a)^{\times\times}$ for each $a \in E$ and we define $U = \bigcup_{a \in E} a + W_a \subset E''$ then it is clear that U is an open subset of E'' that contains E. It is also clear that $\tilde{f}$ is locally bounded in U and so $\tilde{f} \in H(U, F)$.

We remark that $H_G^{w^*u}(E'', F)$, $H^{w^*u}(E'', F)$, and $H^{wu}(E, F)$ with the pointwise multiplication are algebras if F is an algebra, and we can state the following corollaries.

Corollary 11. *Let E be a locally convex space and let F be a complete locally convex space with a structure of algebra. The isomorphism $T: H^{wu}(E, F) \to H_G^{w^*u}(E'', F)$ defined by Theorem* 8 *satisfies $T(f \cdot g) = Tf \cdot Tg$ for all $f, g \in H^{wu}(E, F)$.*

Proof. It is a consequence of the unicity of the extension.

Corollary 12. *Let E be a locally convex space and let F be a complete locally convex space. Let G be a locally convex space such that $E \subset G$ and there exists $S: G \to E''$ linear, continuous with $S|E = \mathrm{id}_E$. Then every $f \in H^{wu}(E, F)$ has an extension $\tilde{f} \in H^{wu}(G, F)$.*

Proof. It is enough to define $\tilde{f} := (Tf) \circ S$.

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References

1. R. Aron and P. Berner, *A Hahn-Banach extension theorem for analytic mappings*, Bull. Soc. Math. France **106** (1978), 3–24.
2. P. Boland, *Holomorphic functions on nuclear spaces*, Trans. Amer. Math. Soc. **209** (1975), 275–281.
3. S. Dineen, *Holomorphically complete locally convex topological vector spaces*, Séminaire Pierre Lelong 1971–72, Lecture Notes in Math., vol. 332, Springer, Berlin, 1973, pp. 77–111.
4. ——, *Complex analysis in locally convex spaces*, North-Holland Math. Stud., vol. 57, North-Holland, Amsterdam, 1981.
5. J. Horváth, *Topological vector spaces and distributions*, vol. I, Addison-Wesley, Reading, MA, 1966.
6. J. Jarchow, *Locally convex spaces*, Teubner, Stuttgart, 1981.
7. R. Meise and D. Vogt, *Counterexamples in holomorphic functions on nuclear Fréchet spaces*, Math. Z. **182** (1983), 167–177.
8. L. A. Moraes, *The Hahn-Banach extension theorem for some spaces of n-homogeneous polynomials*, Functional Analysis: Surveys and Recent Results III (K. D. Bierstedt and B. Fuchsteiner, eds.), North-Holland Math. Stud., vol. 90, North-Holland, Amsterdam, 1984, pp. 265–274.
9. ——, *A Hahn-Banach extension theorem for some holomorphic functions*, Complex Analysis, Functional Analysis and Approximation Theory (J. Mujica, ed.), North-Holland Math. Stud., vol. 125, North-Holland, Amsterdam, 1986, pp. 205–220.
10. ——, *Quotients of spaces of holomorphic functions on Banach spaces*, Proc. Roy. Irish Acad. Sect. A **87** (1987), 181–186.

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