LARGE TIME BEHAVIOR OF THE HEAT KERNEL:
ON A THEOREM OF CHAVEL AND KARP

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Abstract. We show that a theorem of Chavel and Karp follows from the spectral theorem and elliptic regularity.

Recently Chavel and Karp [1] proved the following:

Let \( M \) be a noncompact Riemannian manifold with Laplace-Beltrami operator \( \Delta \) acting on functions on \( M \), \( \lambda =: \lambda(M) \) the bottom of \( \text{spec}(-\Delta) \), and attendant minimal positive heat kernel \( p(x, y, t) \) (where \( (x, y, t) \) is an element of \( M \times M \times (0, +\infty) \)).

Theorem. For all \( x, y \) in \( M \) we have the existence of the limit

\[
\lim_{t \to +\infty} e^{\lambda t} p(x, y, t) =: \mathcal{F}(x, y),
\]

for which we have the following alternative:

Either \( \mathcal{F} \) vanishes identically on all of \( M \times M \), in which case \( \lambda \) possesses no \( L^2 \) eigenfunctions, or \( \mathcal{F} \) is strictly positive on all of \( M \times M \), in which case \( \lambda \) possesses a positive normalized \( L^2 \) eigenfunction \( \phi \) (normalized in the sense that its \( L^2 \) norm is equal to 1) for which

\[
\lim_{t \to +\infty} e^{\lambda t} p(x, y, t) = \phi(x)\phi(y)
\]

locally uniformly on all of \( M \times M \).

Our goal here is to show that this result is essentially an immediate consequence of the spectral theorem and elliptic regularity.

The following well-known lemma follows directly from the spectral theorem and the Lebesgue monotone convergence theorem.

Lemma. Let \( A \) be a selfadjoint operator and let \( f(x, t) \) be a measurable function on \( \sigma(A) \times [0, \infty) \) so that \( f(x, \cdot) \) is monotone for each fixed \( x \) and \( f(x, \infty) = \inf_{t \geq 0} f(x, t) = \lim_{t \to \infty} f(x, t) \). Then, \( s \)-\( \lim_{t \to \infty} f(A, t) = f(A, \infty) \).

For \( t < \infty \), let \( f(x, t) = e^{-t(x-\lambda)} \) and let \( f(x, \infty) = \delta_\lambda(x) \), the characteristic function of \( \{\lambda\} \). Then \( f(-\Delta, \infty) \) is the projection \( P \) onto the space \( S \) of
all $L^2$ eigenfunctions with eigenvalue $\lambda$. Since $p(x, y, t)$ is strictly positive, the Perron-Frobenius theorem (see [2, §XIII.12]) implies that either $S = \{0\}$ or is one-dimensional with a unique element $\varphi$ so that $\varphi(x) > 0$ and $\|\varphi\|_2 = 1$. Thus, either $f(-\Delta, \infty) = 0$ or $f(-\Delta, \infty) = (\varphi, \cdot)\varphi$ as operators.

Equation (1) therefore holds from the lemma if convergence is intended in the $L^2$ sense. To turn this into pointwise convergence (even local $C^\infty$), we need only appeal to elliptic regularity.

By elliptic regularity, $C^\infty(H) \equiv \bigcap_n D(\Delta^n) \supset \text{Ran}(e^{t\Delta})$ consists of $C^\infty$ functions. Thus, $f \mapsto (e^{+\Delta}f)(x)$ is a bounded functional on $L^2$. By duality $g_x(y) \equiv (e^{(\Delta+\lambda)})(x, y)$ is in $L^2$. Thus, by the strong $L^2$ convergence and the semigroup property,

$$e^{\lambda(t+2)}p(x, y, t+2) = \int g_x(z)e^{\lambda t}p(z, w, t)g_y(w)\,dt\,dw$$

converges to $(g_x, Pg_y) = P(x, y)$. This proves the theorem.

We close with several remarks:

1. Since elliptic regularity implies that $C^\infty(H)$ consists of $C^\infty$ functions, it is not hard to see that the convergence is in the $C^\infty$ topology.

2. We did not provide a proof of the last statement in the main theorem of [1] that $\lim_{x \to \infty} \varphi(x) = 0$ if $M$ is noncompact Riemannian with bounded geometry. This should follow by a general subsolution estimate that bounded geometry implies that

$$|\varphi(x)| \leq c \int_{\rho(x, y) \leq 1} |\varphi(y)| \,dy.$$

3. By the proof, the operators $A_t = e^{\lambda t}e^{\Delta t}$ are monotone decreasing in $t$. This implies that $(\delta_x, A\delta_x) = A(x, x)$ is monotone as noted by Chavel-Karp but also that $A(x, x) + A(y, y) + 2A(x, y) = (\delta_x \pm \delta_y, A(\delta_x \pm \delta_y))$ is monotone, providing a direct proof of pointwise convergence.

REFERENCES


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