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A MODULE INDUCED FROM A WHITTAKER MODULE

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ABSTRACT. In an earlier paper [*On modules induced from Whittaker modules*, J. Algebra **96** (1985)] we constructed a class of induced modules, over a finite-dimensional semisimple Lie algebra, which includes the Verma modules of Verma [*Structure of certain induced representations of complex semisimple Lie algebras*, Bull. Amer. Math. Soc. **74** (1968)] and the irreducible Whittaker modules of Kostant [*On Whittaker vectors and representation theory*, Invent. Math. **48** (1978)]. We proved that every module in this class has finite length and is irreducible most of the time. In this article we present a concrete example of this construction, over $\mathrm{sl}(3, C)$, showing that proper submodules can exist when the induced module is not a Verma module.

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. Let $\mathfrak{h}$ be a Cartan subalgebra and let $\mathfrak{n}$ be the sum of the positive root spaces corresponding to a suitable fundamental set of roots. A family of $\mathfrak{g}$-modules $M_{\Omega,c}$ was constructed in [2], where c is a Lie algebra homomorphism taking $\mathfrak{n}$ into $\mathbb{C}$ and Ω is a central character for a certain reductive subalgebra of $\mathfrak{g}$ determined by c. This family includes both the Verma modules of [3] and the irreducible Whittaker modules of [1]. It was shown in [2] that each $M_{\Omega,c}$ has a unique irreducible quotient and that every irreducible $\mathfrak{n}$-finite $\mathfrak{g}$-module is isomorphic to one of these quotients. It was shown in Theorem 2.15 of [2] that $M_{\Omega,c}$ is itself irreducible for most choices of c and Ω. The proof of this assertion was abstract and did not show conclusively that $M_{\Omega,c}$ could have proper submodules in the important case where $M_{\Omega,c}$ is neither a Verma module nor an irreducible Whittaker module. We present a concrete example of this construction here for $\mathfrak{g} = \mathrm{sl}(3, \mathbb{C})$. We derive an explicit equation involving Ω and c that can be used to determine whether $M_{\Omega,c}$ has proper submodules. We show that proper submodules can exist and display explicit vectors that generate them.

We fix $\mathfrak{g} = \mathrm{sl}(3, \mathbb{C})$, the Lie algebra of 3×3 matrices over $\mathbb{C}$ having trace zero, and let $x_1 = e_{12}$, $y_1 = e_{21}$, $h_1 = e_{11} - e_{22}$, $x_2 = e_{23}$, $y_2 = e_{32}$, $h_2 = e_{22} - e_{33}$, $x_3 = e_{13}$, $y_3 = e_{31}$, $h_3 = e_{11} - e_{33} = h_1 + h_2$, and $z = e_{11} + e_{22} - 2e_{33} = h_1 + 2h_2$. Then $y_3, y_2, y_1, h_1, h_2, x_1, x_2$, and x_3 form a basis for $\mathfrak{g}$ over $\mathbb{C}$ and the Lie commutator operation is as shown in Table 1. Using this table and induction on k, we may verify the following lemma.

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TABLE 1. Lie commutator for $\mathrm{sl}(3, \mathbb{C})$

		w							
$[v, w]$		y_3	y_2	y_1	h_1	h_2	x_1	x_2	x_3
	y_3	0	0	0	y_3	y_3	y_2	$-y_1$	x_3
	y_2	0	0	y_3	$-y_2$	$2y_2$	0	$-h_2$	$-x_1$
	y_1	0	$-y_3$	0	$2y_1$	$-y_1$	$-h_1$	0	x_2
v	h_1	$-y_3$	y_2	$-2y_1$	0	0	$2x_1$	$-x_2$	x_3
	h_2	$-y_3$	$-2y_2$	y_1	0	0	$-x_1$	$2x_2$	x_3
	x_1	$-y_2$	0	h_1	$-2x_1$	x_1	0	x_3	0
	x_2	y_1	h_2	0	x_2	$-2x_2$	$-x_3$	0	0
	x_3	h_3	x_1	$-x_2$	$-x_3$	$-x_3$	0	0	0

Lemma 1. *The following equations hold in the universal enveloping algebra* $U(\mathfrak{g})$ *of* $\mathfrak{g}$ *for every positive integer* k.

(1) $$x_1 y_2^k = y_2^k x_1 .$$
(2) $$x_1 h_1^k = (h_1 - 2)^k x_1 .$$
(3) $$y_1 h_1^k = (h_1 + 2)^k y_1 .$$
(4) $$x_2 y_3^k = y_3^k x_2 + k y_3^{k-1} y_1 .$$
(5) $$x_3 y_2^k = y_2^k x_3 + k y_2^{k-1} x_1 .$$
(6) $$x_1 y_3^k = y_3^k x_1 - k y_3^{k-1} y_2 .$$
(7) $$y_1 y_2^k = y_2^k y_1 - k y_3 y_2^{k-1} .$$
(8) $$h_3 y_2^k = y_2^k h_3 - k y_2^k .$$
(9) $$x_2 y_2^k = y_2^k x_2 + k y_2^{k-1} h_2 - k(k-1) y_2^{k-1} .$$
(10) $$x_3 y_3^k = y_3^k x_3 + k y_3^{k-1} h_3 - k(k-1) y_3^{k-1} .$$

$\mathfrak{g}$ has the Cartan subalgebra $\mathfrak{h} = \mathbb{C}h_1 + \mathbb{C}h_2$. Define $\rho_{ij} \in \mathfrak{h}^*$ by $\rho_{ij}(a_1 e_{11} + a_2 e_{22} + a_3 e_{33}) = a_i - a_j$. Then $\rho_{12}, \rho_{23}, \rho_{13}, \rho_{21}, \rho_{32}, \rho_{31}$ are the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ having root vectors $x_1, x_2, x_3, y_1, y_2, y_3$, respectively. $\Delta = \{\rho_{12}, \rho_{23}\}$ is a fundamental set of roots and $\mathfrak{n} = \mathbb{C}x_1 + \mathbb{C}x_2 + \mathbb{C}x_3$ is the corresponding sum of the positive root spaces.

Now we fix a nonzero complex number α and consider the Lie algebra homomorphism $c: \mathfrak{n} \to \mathbb{C}$ for which $c(x_1) = \alpha$, $c(x_2) = 0$, and $c(x_3) = 0$. This homomorphism is singular, in the sense of [1], since it annihilates the fundamental root vector x_2, but it is not identically zero. As in Proposition 1.7 of [2], $\mathfrak{l}$ is the reductive subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$, the fundamental root spaces on which c does not vanish, and the corresponding negative root spaces. In our example $\mathfrak{l} = \mathbb{C}y_1 + \mathbb{C}h_1 + \mathbb{C}h_2 + \mathbb{C}x_1$. Letting $z = h_1 + 2h_2$, we have the decomposition $\mathfrak{l} = \mathfrak{s} \oplus \mathfrak{z}$ where $\mathfrak{s} = \mathbb{C}y_1 + \mathbb{C}h_1 + \mathbb{C}x_1$ is isomorphic to $\mathrm{sl}(2, \mathbb{C})$ and $\mathfrak{z} = \mathbb{C}z$ is the center of $\mathfrak{l}$. We also have $\mathfrak{g} = \overline{\mathfrak{m}} + \mathfrak{l} + \mathfrak{m}$ where $\mathfrak{m} = \mathbb{C}x_2 + \mathbb{C}x_3$

and $\overline{\mathfrak{m}} = \mathbb{C}y_3 + \mathbb{C}y_2$ are nilpotent (in this case abelian) subalgebras of $\mathfrak{g}$. We let $\mathfrak{p}$ denote the parabolic subalgebra $\mathfrak{l} + \mathfrak{m}$.

We let $Z(\mathfrak{s})$ and $Z(\mathfrak{l})$ denote the centers of the enveloping algebras $U(\mathfrak{s})$ and $U(\mathfrak{l})$, respectively. Then $Z(\mathfrak{l})$ is isomorphic to $Z(\mathfrak{s}) \otimes U(\mathfrak{z})$ by the PBW Theorem. $Z(\mathfrak{s})$ and $U(\mathfrak{z})$ are each polynomial algebras generated by the indeterminates $w = 4y_1x_1 + h_1^2 + 2h_1$ and z, respectively. It follows that $Z(\mathfrak{l})$ is a polynomial algebra generated by the two indeterminates w and z.

Given a central character $\Omega\colon Z(\mathfrak{l}) \to \mathbb{C}$, we now construct the induced module $M_{\Omega,c}$ as in Proposition 2.4 of [2]. Evidently Ω is determined by the parameters $\beta = \Omega(z)$ and $\gamma = \Omega(w)$. We begin by constructing the irreducible Whittaker module $Y_{\overline{\Omega},\overline{c}}$ over $\mathfrak{s}$, where $\overline{\Omega}$ is the restriction of Ω to $Z(\mathfrak{s})$ and $\overline{c}$ is the restriction of c to $\mathbb{C}x_1$. This module, constructed formally in Theorem 3.6.1 of [1], is the unique $\mathfrak{s}$-module having central character $\overline{\Omega}$ and generated by a nonzero vector v_0 for which $x_1v_0 = \alpha v_0$. The following lemma provides a useful basis for this module.

Lemma 2. *The $h_1^k v_0$, for $k \in \mathbb{N}$, form a basis for $Y_{\overline{\Omega},\overline{c}}$ over $\mathbb{C}$.*

Proof. To show that this set spans $Y_{\overline{\Omega},\overline{c}}$ it suffices to show that it is stable under the action of $\mathfrak{s}$. Since

$$\gamma v_0 = w v_0 = (4y_1x_1 + h_1^2 + 2h_1)v_0 = 4\alpha y_1 v_0 + h_1^2 v_0 + 2h_1 v_0,$$

it follows that

$$y_1 v_0 = \frac{1}{4\alpha}(\gamma - 2h_1 - h_1^2)v_0.$$

It follows easily from (2) and (3) of Lemma 1 that

$$x_1 h_1^k v_0 = (h_1 - 2)^k x_1 v_0 = \alpha(h_1 - 2)^k v_0 \tag{11}$$

and

$$y_1 h_1^k v_0 = (h_1 + 2)^k y_1 v_0 = \frac{1}{4\alpha}(h_1 + 2)^k(\gamma - 2h_1 - h_1^2)v_0 \tag{12}$$

for each nonnegative integer k so the span of the $h_1^k v_0$ is $\mathfrak{s}$-stable. Finally, the $h_1^k v_0$ are linearly independent; otherwise, they would span a finite-dimensional $\mathfrak{s}$-module and we would have $x_1^n v_0 = 0$ for suitably large n. This is impossible since $x_1^n v_0 = \alpha^n v_0 \neq 0$. □

We extend the action of $\mathfrak{s}$ on $Y_{\overline{\Omega},\overline{c}}$ to an action of $\mathfrak{p}$ by requiring that $zv = \beta v$, $x_2 v = 0$, $x_3 v = 0$ for each $v \in Y_{\overline{\Omega},\overline{c}}$. $M_{\Omega,c}$ is then the induced $\mathfrak{g}$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} Y_{\overline{\Omega},\overline{c}}$. The following lemma, which parallels Proposition 2.4 of [2], lists some basic properties of $M_{\Omega,c}$.

Lemma 3. *Properties of $M_{\Omega,c}$.*

(a) *$M_{\Omega,c}$ is generated as a $\mathfrak{g}$-module by $1 \otimes v_0$. From here on v_0 denotes this vector.*

(b) *The vectors $y_3^{k_3} y_2^{k_2} h_1^{k_1} v_0$, for $k_1, k_2, k_3 \in \mathbb{N}$, form a basis for $M_{\Omega,c}$ over $\mathbb{C}$.*

(c) *Given $\lambda \in \mathbb{C}$, let $M_{\Omega,c}^{\lambda} = \{v \in M_{\Omega,c} | zv = \lambda v\}$. Then each $M_{\Omega,c}^{\lambda}$ is $\mathfrak{l}$-stable, $y_2 M_{\Omega,c}^{\lambda} \subset M_{\Omega,c}^{\lambda-3}$, $y_3 M_{\Omega,c}^{\lambda} \subset M_{\Omega,c}^{\lambda-3}$, $x_2 M_{\Omega,c}^{\lambda} \subset M_{\Omega,c}^{\lambda+3}$, $x_3 M_{\Omega,c}^{\lambda} \subset M_{\Omega,c}^{\lambda+3}$, and $y_3^{k_3} y_2^{k_2} h_1^{k_1} v_0 \in M_{\Omega,c}^{\beta-3k_2-3k_3}$.*

(d) $\mathfrak{z}$ *acts semisimply on* $M_{\Omega,c}$, $M_{\Omega,c} = \bigoplus_{k=0}^{\infty} M_{\Omega,c}^{\beta-3k}$, *and* $M_{\Omega,c}^{\beta} = U(\mathfrak{l})v_0$ *is irreducible as an* $\mathfrak{l}$*-module.*

Proof. (a) follows directly from the construction. (b) follows from Lemma 2 since the $y_3^{k_3}y_2^{k_2}$ from a basis for $U(\mathfrak{g})$ as a free right $U(\mathfrak{p})$-module by the PBW theorem. (c) is easily verified. (d) follows from (c) and from the fact that $Y_{\overline{\Omega},\overline{c}}$ is an irreducible $\mathfrak{s}$-module. □

We now wish to determine when $M_{\Omega,c}$ contains proper submodules. As in [2], we shall call a nonzero vector v in $M_{\Omega,c}$ for which $\mathfrak{m}v = 0$ a maximal vector. The following lemma asserts that it suffices to seek maximal vectors.

Lemma 4. $M_{\Omega,c}$ *contains a proper submodule iff there is a positive integer* n *such that* $M_{\Omega,c}^{\beta-3n}$ *contains a maximal vector.*

Proof. Suppose that $M_{\Omega,c}^{\beta-3n}$ contains a maximal vector v. Since $U(\mathfrak{g}) = U(\overline{\mathfrak{m}})U(\mathfrak{l})U(\mathfrak{m})$, by the PBW theorem, we have $U(\mathfrak{g})v = U(\overline{\mathfrak{m}})U(\mathfrak{l})v$. It now follows from Lemma 3(c) that $U(\mathfrak{g})v \subset \sum_{k\geq n} M_{\Omega,c}^{\beta-3k}$, so $U(\mathfrak{g})v$ is a proper submodule of $M_{\Omega,c}$. Now suppose that N is a proper submodule of $M_{\Omega,c}$. Since $\mathfrak{z}$ acts semisimply on $M_{\Omega,c}$, we have $N = \sum_{k=0}^{\infty} N \cap M_{\Omega,c}^{\beta-3k}$. Let n be the smallest k for which $N \cap M_{\Omega,c}^{\beta-3k}$ is nontrivial. Since any nonzero vector in $M_{\Omega,c}^{\beta}$ generates $M_{\Omega,c}$, n must be positive. It now follows from Lemma 3(c) and the minimality of n that any nonzero vector in $N \cap M_{\Omega,c}^{\beta-3n}$ is maximal. □

It follows from Lemma 3 that there is a bijection between vectors $\hat{v} = \sum_{i=0}^{n} y_3^{n-i}y_2^{i}P_i(h_i)v_0$ in $M_{\Omega,c}^{\beta-3n}$ and sequences of polynomials $P_0(T)$, $P_1(T)$, $\dots$, $P_n(T)$ from $\mathbb{C}[T]$. The following lemma gives the action of x_1, x_2, and x_3 on the typical vector $\hat{v}$ of $M_{\Omega,c}^{\beta-3n}$.

Lemma 5. *Let* n *be a positive integer and let* $\hat{v} = \sum_{i=0}^{n} y_3^{n-i}y_2^{i}P_i(h_1)v_0$ *be a typical vector in* $M_{\Omega,c}^{\beta-3n}$. *Then*

(a)

$$x_1\hat{v} = \alpha y_3^n P_0(h_1 - 2)v_0 + \sum_{i=1}^{n} y_3^{n-i}y_2^{i}(\alpha P_i(h_1 - 2) - (n - i + 1)P_{i-1}(h_1))v_0;$$

(b)

$$x_2\hat{v} = \sum_{i=1}^{n} y_3^{n-i}y_2^{i-1}\left[\frac{1}{4\alpha}(n - i + 1)P_{i-1}(h_1 + 2)(\gamma - 2h_1 - h_1^2) + iP_i(h_1)\left(\frac{1}{2}\beta - \frac{1}{2}h_1 - n + 1\right)\right]v_0;$$

(c)

$$x_3\hat{v} = \sum_{i=1}^{n} y_3^{n-i}y_2^{i-1}\left[i\alpha P_i(h_1 - 2) + (n - i + 1)P_{i-1}(h_1) \times \left(\frac{1}{2}\beta + \frac{1}{2}h_1 - n + 1\right)\right]v_0.$$

Proof. (a) follows from direct computation using (6) and (1) of Lemma 1 and (11) from the proof of Lemma 2. (b) follows from (4), (9), and (7) of Lemma 1, (12) from the proof of Lemma 2, $\mathfrak{m}U(l)v_0 = 0$, and $h_2 v_0 = (\frac{1}{2}z - \frac{1}{2}h_1)v_0 = (\frac{1}{2}\beta - \frac{1}{2}h_1)v_0$. (c) follows from (10), (5), and (8) of Lemma 1, (11) from the proof of Lemma 2, $\mathfrak{m}U(\mathfrak{l})v_0 = 0$, and $h_3 v_0 = (\frac{1}{2}z + \frac{1}{2}h_1)v_0 = (\frac{1}{2}\beta + \frac{1}{2}h_1)v_0$. □

The following proposition states when $M^{\beta-3n}_{\Omega,c}$ contains a maximal vector.

Proposition 1. *Let n be a positive integer. Then $M^{\beta-3n}_{\Omega,c}$ contains a maximal vector iff $\beta = \Omega(z)$ and $\gamma = \Omega(w)$ satisfy the equation*

$$(2n - \beta - 3)^2 = 1 + \gamma. \tag{13}$$

Proof. Let $\hat{v} = \sum_{i=0}^{n} y_3^{n-i} y_2^i P_i(h_1)$ be a typical vector in $M^{\beta-3n}_{\Omega,c}$. It follows from Lemma 5(c) that $x_3\hat{v} = 0$ iff the equations $i\alpha P_i(T-2) + (n-i+1)P_{i-1}(T)(\frac{1}{2}\beta + \frac{1}{2}T - n + 1) = 0$ hold in $\mathbb{C}[T]$ for $i = 1, 2, \dots, n$. Applying the algebra automorphism of $\mathbb{C}[T]$ taking T to $T+2$, we see that this is the case iff

$$i\alpha P_i(T) + (n-i+1)P_{i-1}(T+2)(\tfrac{1}{2}\beta + \tfrac{1}{2}T - n + 2) = 0 \tag{14}$$

in $\mathbb{C}[T]$ for $i = 1, 2, \dots, n$. Nonzero $\hat{v}$ in $M^{\beta-3n}_{\Omega,c}$ for which $x_3\hat{v} = 0$ can always be found by choosing an arbitrary nonzero polynomial $P_0(T)$ and then generating $P_1(T), P_2(T), \dots, P_n(T)$ from (14). From Lemma 5(b) we have $x_2\hat{v} = 0$ iff

$$\begin{aligned}&\frac{1}{4\alpha}(n-i+1)P_{i-1}(T+2)(\gamma - 2T - T^2)\\ &\qquad + iP_i(T)\left(\frac{1}{2}\beta - \frac{1}{2}T - n + 1\right) = 0\end{aligned} \tag{15}$$

in $\mathbb{C}[T]$ for $i = 1, 2, \dots, n$. Thus $\hat{v}$ is maximal iff $P_0(T) \neq 0$ and equations (14) and (15) hold. Using equations (14) to substitute for $P_i(T)$ in terms of $P_{i-1}(T+2)$ in equations (15), we have that $\hat{v}$ is maximal iff $P_0(T) \neq 0$, equations (14) hold, and

$$\begin{aligned}&\frac{1}{4\alpha}(n-i+1)P_{i-1}(T+2)(\gamma - 2T - T^2)\\ &\quad + \frac{1}{\alpha}(n-i+1)P_{i-1}(T+2)\left(n - \frac{1}{2}\beta - \frac{1}{2}T - 2\right)\left(\frac{1}{2}\beta - \frac{1}{2}T - n + 1\right)\\ &= 0\end{aligned}$$

in $\mathbb{C}[T]$ for $i = 1, 2, \dots, n$. It follows from this that $\hat{v}$ is maximal iff $P_0(T) \neq 0$, equations (14) hold, and

$$\begin{aligned}0 &= \gamma - 2T - T^2 + 4\left(n - \frac{1}{2}\beta - \frac{1}{2}T - 2\right)\left(\frac{1}{2}\beta - \frac{1}{2}T - n + 1\right)\\ &= \gamma - 4n^2 + 4\beta n + 12n - \beta^2 - 6\beta - 8\\ &= 1 + \gamma - 4n^2 + 4(\beta+3)n - (\beta+3)^2\\ &= 1 + \gamma - (2n - \beta - 3)^2. \quad \square\end{aligned}$$

As in [2], a c-vector is a vector $v \in M_{\Omega,c}$ such that $xv = c(x)v$ for all $x \in \mathfrak{n}$. Clearly, every nonzero c-vector is maximal and a maximal vector v is a c-vector iff $x_1 v = \alpha v$. The following proposition determines the c-vectors of $M_{\Omega,c}^{\beta-3n}$.

Proposition 2. *Let n be a positive integer. Then $M_{\Omega,c}^{\beta-3n}$ contains a nonzero c-vector iff it contains a maximal vector. In this case the space of c-vectors is the one-dimensional vector space spanned by $\hat{v} = \sum_{i=0}^{n} y_3^{n-i} y_2^i P_i(h_1) v_0$ where $P_0(T) = 1$, $\Lambda(T) = 2n - \beta - T - 4$, and*

$$P_i(T) = \frac{n(n-1)\cdots(n-i+1)}{(2\alpha)(4\alpha)\cdots(2i\alpha)} \Lambda(T)\Lambda(T+2)\cdots\Lambda(T+2i-2)$$

for $i = 1, 2, \ldots, n$.

Proof. Since every nonzero c-vector is maximal, one direction is obvious. Assume that $M_{\Omega,c}^{\beta-3n}$ has a maximal vector. Then (13) holds and any vector $v = \sum_{i=0}^{n} y_3^{n-i} y_2^i P_i(h_1) v_0$ for which $P_0(T) \neq 0$ and equations (14) are satisfied is a maximal vector. It follows from Lemma 5(a) that such a vector is a c-vector iff $\alpha P_0(T-2) = \alpha P_0(T)$ and

$$\alpha P_i(T-2) - (n-i+1) P_{i-1}(T) = \alpha P_i(T) \tag{16}$$

holds in $\mathbb{C}[T]$ for $i = 1, 2, \ldots, n$. The first equation holds iff $P_0(T)$ is a constant polynomial, in which case equation (14) holds iff v is a scalar multiple of the vector $\hat{v}$ given in the hypothesis. It is easily verified that $\hat{v}$ satisfies equations (16) and thus is a c-vector. □

The following corollary states when $M_{\Omega,c}$ has proper submodules.

Corollary 1. *$M_{\Omega,c}$ has a proper submodule iff $(2n - \beta - 3)^2 = 1 + \gamma$ holds for some positive integer n. In this case $U(\mathfrak{g})\hat{v}$ is a proper submodule, where $\hat{v}$ is the vector given in the hypothesis of Proposition 2.*

Proof. This follows immediately from Lemma 4, Proposition 1, and Proposition 2. □

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