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INVERSIONS OF HERMITE SEMIGROUP

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ABSTRACT. Let $\{e^{-cH}|c \geq 0\}$ be the Hermite semigroup on the real line $\mathbb{R}$. Then a representation is constructed for inversions of the semigroup, and it gives a representation of e^{-cH} for $c < 0$. Moreover, some characterizations of the domain in which, for $c < 0$, e^{-cH} is well defined are examined.

1. INTRODUCTION

We let μ be the Gaussian measure $\frac{1}{\sqrt{\pi}}e^{-x^2}dx$ on $\mathbb{R}$. Then the family of the normalized Hermite polynomials

$$h_n(x) = \frac{1}{\sqrt{2^n n!}}e^{x^2}\frac{d^n}{dx^n}e^{-x^2}, \qquad n = 0, 1, 2, \dots,$$

is a complete orthonormal system in $L^2(\mu)$. For any $f \in L^2(\mu)$, let $f(x) = \sum_{n=0}^{\infty} a_n h_n(x)$. If c is nonnegative, then the series $\sum_{n=0}^{\infty} a_n e^{-cn} h_n(x)$ converges in $L^2(\mu)$. Hence we can define the linear operator e^{-cH} on $L^2(\mu)$ by

$$[e^{-cH}f](x) = \sum_{n=0}^{\infty} a_n e^{-cn} h_n(x), \tag{1.1}$$

and the operator norm of e^{-cH} is 1. The Hermite semigroup on $\mathbb{R}$ means the family $\{e^{-cH}|c \geq 0\}$. More generally, for every complex number c with $\Re c > 0$ or $c = 0$, we shall consider the operator e^{-cH}, which is examined in several papers (for instance, see [4, 10]). For $\Re c > 0$, the operator e^{-cH} is represented by

$$[e^{-cH}f](x) = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} p_c(x-\xi)f(\xi)e^{-\xi^2}\,d\xi, \qquad f \in L^2(\mu),\ x \in \mathbb{R}, \tag{1.2}$$

where the integral kernel

$$p_c(x-\xi) = \sum_{n=0}^{\infty} e^{-cn}h_n(x)h_n(\xi) = (1-e^{-2c})^{-1/2}\exp\left\{\xi^2 - \frac{(\xi - e^{-c}x)^2}{1-e^{-2c}}\right\}.$$

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In this paper, for any c with $\Re c > 0$ we shall give a representation for the inverse of the operator e^{-cH} in terms of integrals. Our argument for its representation is directly related to an extension of the operator e^{-cH} for $\Re c < 0$. When $\Re c < 0$, it is obvious that the domain in which the operator e^{-cH} is well defined is properly contained in $\mathrm{L}^2(\mu)$. We let $\mathscr{D}(e^{-cH}) = \{f \in \mathrm{L}^2(\mu) | e^{-cH} f \in \mathrm{L}^2(\mu)\}$. Then we give two characterizations of the members in $\mathscr{D}(e^{-cH})$ in terms of analytic extensibility of members in $\mathrm{L}^2(\mu)$ as entire functions.

We shall use the general method for integral transforms ([8] or [9, p. 82]) and some ideas for best approximation oriented in [1] on reproducing kernel Hilbert spaces.

2. Inverse of e^{-cH}

For any fixed complex number c with $\Re c > 0$, applying the dominated convergence theorem and Morera's theorem to the integral representation (1.2), we see that every member in the range of the operator e^{-cH} can be analytically extended to the complex plane $\mathbb{C}$. Hence we shall consider the operator e^{-cH} as the linear operator of $\mathrm{L}^2(\mu)$ into an entire function space. Then, following the method of characterizing the ranges of integral transforms ([8] or [9, p. 82]), we obtain

Theorem 1.1. *For any c with $\Re c > 0$, the range of $\mathrm{L}^2(\mu)$ under the operator e^{-cH} coincides with the Hilbert space consisting of entire functions with finite norms*

$$\|g\|_c^2 = \frac{2|w|^2}{\pi\sqrt{1-|w|^4}} \cdot \iint_{\mathbb{C}} |g(z)|^2 \exp\left\{-2|w|^2\left(\frac{x^2}{1+|w|^2} + \frac{y^2}{1-|w|^2}\right)\right\} dx\,dy \tag{2.1}$$

where $z = x + iy$ and w denotes e^{-c}. Moreover, the isometrical identity

$$\|e^{-cH} f\|_c^2 = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} |f(x)|^2 e^{-x^2}\, dx, \qquad f \in \mathrm{L}^2(\mu),$$

is valid.

Proof. For any pair $(z, u) \in \mathbb{C} \times \mathbb{C}$, we calculate the complex kernel form

$$\begin{aligned} K(z, \overline{u}; c) &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} p_c(z-\xi)\overline{p_c(u-\overline{\xi})} e^{-\xi^2}\, d\xi \\ &= \frac{1}{\sqrt{\pi(1-w^2)(1-\overline{w}^2)}} \exp\left\{\frac{-w^2 z^2}{1-w^2}\right\} \exp\left\{\frac{-\overline{w}^2\overline{u}^2}{1-\overline{w}^2}\right\} \\ &\quad \cdot \int_{\mathbb{R}} \exp\left\{-\frac{1-|w|^4}{(1-w^2)(1-\overline{w}^2)}\xi^2 \right. \\ &\qquad \left. + \frac{2(1-\overline{w}^2)wz + 2(1-w^2)\overline{w}\overline{u}}{(1-w^2)(1-\overline{w}^2)}\xi\right\} d\xi \\ &= \frac{1}{\sqrt{1-|w|^4}} \exp\left\{\frac{-|w|^4 z^2}{1-|w|^4}\right\} \exp\left\{\frac{-|w|^4\overline{u}^2}{1-|w|^4}\right\} \exp\left\{\frac{2|w|^2 z\overline{u}}{1-|w|^4}\right\}. \end{aligned} \tag{2.2}$$

Since $K(z, \overline{u}; c)$ is a positive matrix on $\mathbb{C}$, it uniquely determines the reproducing kernel Hilbert space H_c admitting the reproducing kernel $K(z, \overline{u}; c)$. From [8] or [9, p. 82], the space H_c is the range of $\mathrm{L}^2(\mu)$ under the operator e^{-cH} and we have the isometrical identity

$$\|e^{-cH}f\|_{H_c}^2 = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} |f(x)|^2 e^{-x^2}\,dx, \qquad f \in \mathrm{L}^2(\mu),$$

because the family $\{p_c(z-\xi) | z \in \mathbb{C}\}$ is complete in $\mathrm{L}^2(\mu)$. Hence it is sufficient to prove that the members in H_c are characterized as entire functions with finite norms (2.1). If $g \in H_c$, then g is expressible in the form

$$g(z) = (1-|w|^4)^{-1/2} \exp\left\{\frac{-|w|^4 z^2}{1-|w|^4}\right\} g_1(z), \tag{2.3}$$

where $z \in \mathbb{C}$ and g_1 is a member in the reproducing kernel Hilbert space $H_{\exp\{2|w|^2 z\overline{u}/(1-|w|^4)\}}$ admitting the reproducing kernel $\exp\{\frac{2|w|^2 z\overline{u}}{1-|w|^4}\}$. Moreover, the following isometrical identity holds:

$$\|g\|_{H_c}^2 = (1-|w|^4)^{-1/2} \|g_1\|^2_{H_{\exp\{2|w|^2 z\overline{u}/(1-|w|^4)\}}}. \tag{2.4}$$

For these contents, see [2, p. 358]. Meanwhile, referring to [3, p. 198], we have

$$\|g_1\|^2_{H_{\exp\{2|w|^2 z\overline{u}/(1-|w|^4)\}}} = \frac{2|w|^2}{\pi(1-|w|^4)} \iint_{\mathbb{C}} |g_1(z)|^2 \exp\left\{-\frac{2|w|^2|z|^2}{1-|w|^4}\right\} dx\,dy.$$

Therefore, from (2.3) and (2.4) our theorem is proved.

For two complex numbers c_1, c_2 with $\Re c_1 > 0$ and $\Re c_2 > 0$, we shall discuss a relation between H_{c_1} and H_{c_2}. Put $g_1 = e^{-c_1 H} f$ and $g_2 = e^{-c_2 H} f$ for some $f \in \mathrm{L}^2(\mu)$. If $\Re c_1 > \Re c_2$, then $g_1 = e^{-c_1 H} f = e^{-(c_1-c_2)H} e^{-c_2 H} f = e^{-(c_1-c_2)H} g_2$, and so we can directly obtain a representation of g_1 in terms of g_2 by using the integral representation (1.2). However, if $\Re c_1 \le \Re c_2$, it is not obvious to represent g_1 in terms of g_2. Hence we are interested in this case. Furthermore, we shall establish the inverse formula of the integral transform (1.2).

Theorem 2.2. *For $\Re c_1 > 0$, $\Re c_2 > 0$, and $f \in \mathrm{L}^2(\mu)$, let $g_1 = e^{-c_1 H} f$ and $g_2 = e^{-c_2 H} f$. Then g_1 is expressible in the form*

$$\begin{aligned} g_1(\xi) = {} & \frac{2|w_2|^2}{\pi\sqrt{(1-|w_2|^4)(1-w_1^2\overline{w_2}^2)}} \exp\left\{\frac{-w_1^2\overline{w_2}^2\xi^2}{1-w_1^2\overline{w_2}^2}\right\} \\ & \cdot \iint_{\mathbb{C}} g_2(z) \exp\left\{\frac{w_1\overline{w_2}\,\overline{z}}{1-w_1^2\overline{w_2}^2}(2\xi - w_1\overline{w_2}\,\overline{z}) \right. \\ & \left. - 2|w_2|^2\left(\frac{x^2}{1+|w_2|^2} + \frac{y^2}{1-|w_2|^2}\right)\right\} dx\,dy, \end{aligned} \tag{2.5}$$

where $w_1 = e^{-c_1}$, $w_2 = e^{-c_2}$, and $g_2(z)$ is the analytic extension of g_2 to $\mathbb{C}$.

Moreover, f *is given by*

$$f(\xi) = \operatorname*{l.i.m.}_{r\uparrow 1} \frac{2|w_2|^2}{\pi\sqrt{(1-|w_2|^4)(1-r^2\overline{w_2}^2)}} \exp\left\{\frac{-r\overline{w_2}^2\xi^2}{1-r^2\overline{w_2}^2}\right\}$$

$$\tag{2.6} \cdot \iint_{\mathbb{C}} g_2(z) \exp\left\{\frac{r\overline{w_2}\,\overline{z}}{1-r^2\overline{w_2}^2}(2\xi - r\overline{w_2}\,\overline{z}) - 2|w_2|^2\left(\frac{x^2}{1+|w_2|^2} + \frac{y^2}{1-|w_2|^2}\right)\right\} dx\, dy.$$

The notation l.i.m. *means the* $\mathrm{L}^2(\mu)$*-convergence.*

Proof. From the definition (1.1) of $e^{-c_2 H}$ and the identity (2.5), the expression (2.6) is obvious. We assume that T_{c_1} is the inverse operator of $e^{-c_1 H}$ from H_{c_1} to $\mathrm{L}^2(\mu)$. In addition, let S_{c_1, c_2} be the linear operator of H_{c_1} into H_{c_2} defined by

$$S_{c_1, c_2} g = e^{-c_2 H} T_{c_1} g, \qquad g \in H_{c_1}.$$

Then we have $S_{c_1, c_2} g_1 = g_2$. Since the operator S_{c_1, c_2} is an isometry of H_{c_1} onto H_{c_2}, the adjoint operator $S^*_{c_1, c_2}$ of S_{c_1, c_2} is the inverse of it. Hence, for $\xi \in \mathbb{R}$ we get the representation

$$g_1(\xi) = [S^*_{c_1, c_2} g_2](\xi) = (S^*_{c_1, c_2} g_2, K(\cdot, \xi; c_1))_{H_{c_1}} = (g_2, S_{c_1, c_2} K(\cdot, \xi; c_1))_{H_{c_2}}.$$

Meanwhile, for $z \in \mathbb{C}$ the following is valid:

$$\begin{aligned}
&[S_{c_1, c_2} K(\cdot, \xi; c_1)](z) \\
&= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} p_{c_2}(z-x)\overline{p_{c_1}(\xi - x)} e^{-x^2}\, dx \\
&= \frac{1}{\sqrt{\pi(1-\overline{w_1}^2)(1-w_2^2)}} \exp\left\{\frac{-\overline{w_1}^2\xi^2}{1-\overline{w_1}^2}\right\} \exp\left\{\frac{-w_2^2 z^2}{1-w_2^2}\right\} \\
&\quad \cdot \int_{\mathbb{R}} \exp\left\{-\frac{1-\overline{w_1}^2 w_2^2}{(1-\overline{w_1}^2)(1-w_2^2)} x^2 + \frac{2(1-w_2^2)\overline{w_1}\xi + 2(1-\overline{w_1}^2) w_2 z}{(1-\overline{w_1}^2)(1-w_2^2)} x\right\} dx \\
&= \frac{1}{\sqrt{1-\overline{w_1}^2 w_2^2}} \exp\left\{\frac{-\overline{w_1}^2 w_2^2 \xi^2}{1-\overline{w_1}^2\overline{w_2}^2 w}\right\} \\
&\quad \cdot \exp\left\{\frac{-\overline{w_1}^2 w_2^2 z^2}{1-\overline{w_1}^2 w_2^2}\right\} \exp\left\{\frac{2\overline{w_1} w_2 z \xi}{1-\overline{w_1}^2 w_2^2}\right\}.
\end{aligned}$$

Therefore, we obtain the identity (2.5).

In the expression (2.6), r can be taken on a path in $\{0 < |z| < 1\}$ with the terminal point 1, in general.

Remark. For any n, $h_n(z)$ denotes the analytic extension of $h_n(x)$ to $\mathbb{C}$. Then we see that, for $\Re c > 0$, the family $\{e^{-cn} h_n(z) | n = 0, 1, 2, \ldots\}$ is a complete orthonormal system in H_c. Hence, the expression $p_c(z-\xi) = \sum_{n=0}^{\infty} e^{-cn} h_n(z) h_n(\xi)$ and Theorem 1.1 suggest the representation of T_c in the form

$$[T_c g](\xi) = (g\,, p_c(z-\xi))_{H_c}\,, \qquad g \in H_c. \tag{2.7}$$

For fixed ξ, however, the integral in the right-hand side of (2.7) need not converge. In this case, in [3, p. 202; 7; 9, p. 85] the method of taking some exhaustions of $\mathbb{C}$ was considered in a natural way. According to such a general method, we can obtain another representation of (2.6):

$$\begin{aligned} f(\xi) &= \operatorname*{l.\,i.\,m.}_{\sigma\to\infty} \frac{2|w_2|^2}{\pi\sqrt{1-|w_2|^4}} \\ &\quad \cdot \iint_{|z|\le\sigma} g_2(z)\overline{p_{c_2}(z-\xi)} \exp\left\{-2|w_2|^2\left(\frac{x^2}{1+|w_2|^2} + \frac{y^2}{1-|w_2|^2}\right)\right\} dx\,dy \\ &= \operatorname*{l.\,i.\,m.}_{\sigma\to\infty} \frac{2|w_2|^2}{\pi\sqrt{(1-|w_2|^4)(1-\overline{w_2}^2)}} \\ &\quad \cdot \iint_{|z|\le\sigma} g_2(z) \exp\left\{\xi^2 - \frac{(\xi-\overline{w_2}\,\overline{z})^2}{1-\overline{w_2}^2}\right. \\ &\qquad\qquad \left. -\,2|w_2|^2\left(\frac{x^2}{1+|w_2|^2} + \frac{y^2}{1-|w_2|^2}\right)\right\} dx\,dy. \end{aligned}$$

3. Extension of e^{-cH} for $\Re c < 0$

The inverse transform (2.6) means an extension of e^{-cH} for $\Re c < 0$ because $f(x)$ has the representation $\sum_{n=0}^{\infty} a_n e^{-(-c_2)n} h_n(x)$ in $\mathrm{L}^2(\mu)$ when $g_2(x) = \sum_{n=0}^{\infty} a_n h_n(x)$. Hence, for any c with $\Re c < 0$, we define the linear operator e^{-cH} in the form (1.1). Then, in the expression (2.6) replacing w_2 by e^c, we obtain the representation of e^{-cH}. However, since the expression (2.6) requires the analytic extension form of a member in $\mathrm{L}^2(\mu)$, we shall give its representation in terms of real variable.

For any fixed c with $\Re c < 0$, we first establish two characterizations of the members in $\mathscr{D}(e^{-cH})$. Since the family $\{e^{cn}h_n(z) | n = 0, 1, 2, \dots\}$ is a complete orthonormal system in H_{-c}, for a given $f \in \mathrm{L}^2(\mu)$, $f \in \mathscr{D}(e^{-cH})$ if and only if f is almost everywhere equal to the restriction of a member in H_{-c} to $\mathbb{R}$ with respect to μ. In this connection, we recall the interesting result in [5, p. 166]:

For any $\alpha > 0$, let h be an entire function satisfying $\iint_{\mathbb{C}} |h(z)|^2 e^{-y^2/\alpha}\, dx\, dy < \infty$. Then

$$\frac{1}{\sqrt{\pi\alpha}} \iint_{\mathbb{C}} |h(z)|^2 e^{-y^2/\alpha}\, dx\, dy = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \int_{\mathbb{R}} \left|\frac{d^n}{dx^n} h(x)\right|^2 dx. \tag{3.1}$$

Conversely, if $h(x)$ is a C^∞-function on $\mathbb{R}$ with a convergent sum in the right-hand side of (3.1), *then h can be analytically extended to $\mathbb{C}$ and satisfies the identity* (3.1).

This result will contribute to our result in the following way.

Theorem 3.1. *For any c with $\Re c < 0$, let $w = e^c$. If f is a C^∞-function in* $\mathrm{L}^2(\mu)$, *then the following are equivalent*:

(1) f *is a member in* $\mathscr{D}(e^{-cH})$.

(2) *The series*

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1-|w|^4}{4|w|^2}\right)^n \int_{\mathbb{R}} \left| \frac{d^n}{dx^n}\left[f(x)\exp\left\{-\frac{|w|^2x^2}{1+|w|^2}\right\}\right]\right|^2 dx \tag{3.2}$$

converges.

(3) *The integral*

$$\begin{aligned} &\iint_{\mathbb{C}} \left| \int_{\mathbb{R}} f(\xi)\exp\left\{\frac{(2|w|^2z-\xi)\xi}{1-|w|^4}\right\} d\xi\right|^2 \\ &\quad \cdot \exp\left[\frac{-2|w|^4}{(1-|w|^4)(|w|^8+|w|^4+1)}\right. \\ &\qquad \left. \cdot \{(|w|^4+|w|^2+1)x^2 - (|w|^4-|w|^2+1)y^2\}\right] dx\,dy \end{aligned} \tag{3.3}$$

is finite.

Proof. If f is the restriction of a member $\hat{f}$ in H_{-c} to $\mathbb{R}$, then we have

$$\|\hat{f}\|_{-c}^2 = \frac{2|w|^2}{\pi\sqrt{1-|w|^4}} \iint_{\mathbb{C}} \left|\hat{f}(z)\exp\left\{\frac{-|w|^2z^2}{1+|w|^2}\right\}\right|^2 \exp\left\{\frac{-4|w|^2y^2}{1-|w|^4}\right\} dx\,dy. \tag{3.4}$$

Therefore, from the identity (3.1) the two statements (1) and (2) are equivalent.

Next, we prove that f is the restriction of a member in H_{-c} to $\mathbb{R}$ if and only if f satisfies the condition (3). For simplicity, let T be the restriction operator to $\mathbb{R}$. Then T is a bounded operator of H_{-c} into $\mathrm{L}^2(\mu)$ (in fact, the operator norm $\|T\|$ is 1), and the adjoint operator T^* of T has the following representation:

$$[T^*h](z) = (h, TK(\cdot, \overline{z}; -c))_{\mathrm{L}^2(\mu)}, \qquad z \in \mathbb{C},$$

for all $h \in \mathrm{L}^2(\mu)$. Since the range $T(H_{-c})$ is dense in $\mathrm{L}^2(\mu)$, the operator T^* is one-to-one. Hence, f can be extended as the member in H_{-c} if and only if T^*f is contained in the range $T^*T(H_{-c})$. If g is a member in $T^*T(H_{-c})$, then g is expressible in the form

$$\begin{aligned} g(z) &= (T^*TG, K(\cdot, \overline{z}; -c))_{H_{-c}} \\ &= (G, T^*TK(\cdot, \overline{z}; -c))_{H_{-c}}, \qquad z \in \mathbb{C}, \end{aligned} \tag{3.5}$$

for a member G in H_{-c} with $T^*TG = g$. Following a general method in [8] or [9, p. 82], we shall give a characterization of the members in $T^*T(H_{-c})$. From the representation (3.5) of T^*T, we calculate the kernel form

$$\begin{aligned} k(z, \overline{u}; c) &= (T^*TK(\cdot, \overline{u}; -c), T^*TK(\cdot, \overline{z}; -c))_{H_{-c}} \\ &= (TK(\cdot, \overline{u}; -c), TT^*TK(\cdot, \overline{z}; -c))_{\mathrm{L}^2(\mu)}. \end{aligned} \tag{3.6}$$

Meanwhile, for all $z' \in \mathbb{C}$ we have

$$
\begin{aligned}
&[T^*TK(\cdot\,, \overline{z}\,; -c)](z') \\
&\quad = (TK(\cdot\,, \overline{z}\,; -c)\,, TK(\cdot\,, \overline{z}'\,; -c))_{L^2(\mu)} \\
&\quad = \frac{1}{(1-|w|^4)\sqrt{\pi}} \exp\left\{\frac{-|w|^4 z'^2}{1-|w|^4}\right\} \exp\left\{\frac{-|w|^4\overline{z}^2}{1-|w|^4}\right\} \\
&\qquad \cdot \int_{\mathbf{R}} \exp\left\{-\frac{1+|w|^4}{1-|w|^4}\xi^2 + \frac{2|w|^2(\overline{z}+z')}{1-|w|^4}\xi\right\} d\xi \\
&\quad = \frac{1}{\sqrt{1-|w|^8}} \exp\left\{\frac{-|w|^8 z'^2}{1-|w|^8}\right\} \exp\left\{\frac{-|w|^8\overline{z}^2}{1-|w|^8}\right\} \exp\left\{\frac{2|w|^4 z'\overline{z}}{1-|w|^8}\right\}.
\end{aligned} \tag{3.7}
$$

Hence, the desired kernel form is given by

$$
\begin{aligned}
k(z\,, \overline{u}\,; c) &= \frac{1}{\sqrt{\pi(1-|w|^4)(1-|w|^8)}} \exp\left\{\frac{-|w|^8 z^2}{1-|w|^8}\right\} \exp\left\{\frac{-|w|^4\overline{u}^2}{1-|w|^4}\right\} \\
&\quad \cdot \int_{\mathbf{R}} \exp\left\{-\frac{|w|^8+|w|^4+1}{1-|w|^8}\xi^2 + \left(\frac{2|w|^4 z}{1-|w|^8} + \frac{2|w|^2\overline{u}}{1-|w|^4}\right)\xi\right\} d\xi \\
&= \frac{1}{\sqrt{(1-|w|^4)(|w|^8+|w|^4+1)}} \exp\left\{\frac{-|w|^{12} z^2}{(1-|w|^4)(|w|^8+|w|^4+1)}\right\} \\
&\quad \cdot \exp\left\{\frac{-|w|^{12}\overline{u}^2}{(1-|w|^4)(|w|^8+|w|^4+1)}\right\} \\
&\quad \cdot \exp\left\{\frac{2|w|^6 z\overline{u}}{(1-|w|^4)(|w|^8+|w|^4+1)}\right\}.
\end{aligned} \tag{3.8}
$$

Then the range $T^*T(H_{-c})$ coincides with the reproducing kernel Hilbert space $H_{k(c)}$ admitting the reproducing kernel $k(z\,, \overline{u}\,; c)$, and T^*T is an isometry of H_{-c} onto $H_{k(c)}$. Let H be the reproducing kernel Hilbert space determined by the positive matrix $\exp[2|w|^6 z\overline{u}/\{(1-|w|^4)(|w|^8+|w|^4+1)\}]$. As in the proof of Theorem 1.2, we obtain the factorization

$$
\begin{aligned}
g(z) &= \frac{1}{\sqrt{(1-|w|^4)(|w|^8+|w|^4+1)}} \\
&\quad \cdot \exp\left\{\frac{-|w|^{12} z^2}{(1-|w|^4)(|w|^8+|w|^4+1)}\right\} g_1(z)
\end{aligned} \tag{3.9}
$$

for an entire function $g_1 \in H$. So it gives the norm identity

$$
\|g\|^2_{H_{k(c)}} = \frac{1}{\sqrt{(1-|w|^4)(|w|^8+|w|^4+1)}} \|g_1\|^2_H\,,
$$

where

$$
\begin{aligned}
\|g_1\|^2_H &= \frac{2|w|^6}{\pi(1-|w|^4)(|w|^8+|w|^4+1)} \\
&\quad \cdot \iint_{\mathbf{C}} |g_1(z)|^2 \exp\left\{\frac{-2|w|^6|z|^2}{(1-|w|^4)(|w|^8+|w|^4+1)}\right\} dx\,dy.
\end{aligned} \tag{3.10}
$$

Note that H is the totality of entire functions with finite norms (3.10). Hence, the space $H_{k(c)}$ consists of entire functions with finite norms

$$\begin{aligned} \|g\|^2_{H_{k(c)}} &= \frac{2|w|^6}{\pi\sqrt{(1-|w|^4)(|w|^8+|w|^4+1)}} \\ &\cdot \iint_{\mathbf{C}} |g(z)|^2 \exp\left[\frac{-2|w|^2}{(1-|w|^4)(|w|^8+|w|^4+1)}\right. \\ &\left.\cdot\{(1-|w|^6)x^2+(1+|w|^6)y^2\}\right] dx\,dy. \end{aligned} \tag{3.11}$$

Since $T^* f$ is expressible in the form

(3.12)
$$[T^* f](z) = \frac{1}{\sqrt{\pi(1-|w|^4)}} \exp\left\{\frac{-|w|^4 z^2}{1-|w|^4}\right\} \int_{\mathbf{R}} f(\xi) \exp\left\{\frac{(2|w|^2 z-\xi)\xi}{1-|w|^4}\right\} d\xi,$$

our claim has been proved.

When a C^∞-function $f \in L^2(\mu)$ satisfies one of three items in Theorem 3.1, we shall give the representation of the analytic extension $\hat{f}$ in terms of f. By the isometry T^*T of H_{-c} onto $H_{k(c)}$, we have

Theorem 3.2. *Let c, w, and f be as in Theorem 3.1. If f satisfies one of three items in Theorem 3.1 and $\hat{f}$ denotes the analytic extension of f to $\mathbb{C}$, then $\hat{f}$ is represented by*

$$\begin{aligned} \hat{f}(z) &= \frac{2|w|^6}{\pi^{3/2}(1-|w|^4)\sqrt{(1-|w|^8)(|w|^8+|w|^4+1)}} \exp\left\{\frac{-|w|^8 z^2}{1-|w|^8}\right\} \\ &\cdot \iint_{\mathbf{C}} \left[\int_{\mathbf{R}} f(\xi) \exp\left\{\frac{(2|w|^2 Z-\xi)\xi}{1-|w|^4}\right\} d\xi\right] \\ &\cdot \exp\left[\frac{-|w|^4}{1-|w|^8}(Z^2+|w|^4\overline{Z}^2-2\overline{Z}z)\right. \\ &\left.-\frac{2|w|^6\{(1-|w|^6)X^2+(1+|w|^6)Y^2\}}{(1-|w|^4)(|w|^8+|w|^4+1)}\right] dX\,dY, \end{aligned} \tag{3.13}$$

where $Z = X + iY$.

Proof. Let S be the adjoint operator of T^*T from $H_{k(c)}$ to H_{-c}. Then, since T^*T is an isometry of H_{-c} onto $H_{k(c)}$, the operator S is the inverse of T^*T. Hence, $T^*f = T^*TST^*f$ and $f = TST^*f$. Since

$$\begin{aligned} \hat{f}(z) &= [ST^*f](z) = (ST^*f, K(\cdot, \overline{z}; -c))_{H_{-c}} \\ &= (T^*f, T^*TK(\cdot, \overline{z}; -c))_{H_{k(c)}}, \end{aligned}$$

from (3.7), (3.11), and (3.12) we obtain the desired representation (3.13).

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