CONVEX REAL PROJECTIVE STRUCTURES
ON CLOSED SURFACES ARE CLOSED

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Abstract. The deformation space $\mathcal{C}(\Sigma)$ of convex $\mathbb{RP}^2$-structures on a closed surface $\Sigma$ with $\chi(\Sigma) < 0$ is closed in the space $\text{Hom}(\pi, \text{SL}(3, \mathbb{R}))/\text{SL}(3, \mathbb{R})$ of equivalence classes of representations $\pi(\Sigma) \to \text{SL}(3, \mathbb{R})$. Using this fact, we prove Hitchin’s conjecture that the contractible “Teichmüller component” (Lie groups and Teichmüller space, preprint) of $\text{Hom}(\pi, \text{SL}(3, \mathbb{R}))/\text{SL}(3, \mathbb{R})$ precisely equals $\mathcal{C}(\Sigma)$.

Let $\Sigma$ be a closed orientable surface of genus $g > 1$ and $\pi = \pi_1(\Sigma)$ its fundamental group. A convex $\mathbb{RP}^2$-structure on $M$ is a representation (uniformization) of $M$ as a quotient $\Omega/\Gamma$ where $\Omega \subset \mathbb{RP}^2$ is a convex domain and $\Gamma \subset \text{SL}(3, \mathbb{R})$ is a discrete group of collineations of $\mathbb{RP}^2$ acting properly and freely on $\Omega$. (See [5] for basic theory of such structures.) The space of projective equivalence classes of convex $\mathbb{RP}^2$-structures embeds as an open subset in the space of equivalence classes of representations $\pi \to \text{SL}(3, \mathbb{R})$. The purpose of this note is to show that this subset is also closed.

In [7], Hitchin shows that the space of equivalence classes of representations $\pi \to \text{SL}(3, \mathbb{R})$ falls into three connected components: one component $C_1$ consisting of classes of representations for which the associated flat $\mathbb{R}^3$-bundle over $\Sigma$ has nonzero second Stiefel-Whitney class; a component $C_0$ containing the class of the trivial representation; a component $C_1$ diffeomorphic to a cell of dimension $16(g - 1)$, which he calls the “Teichmüller component.” While $C_1$ can be distinguished from $C_0$ and $C_1$ by a topological invariant [3, 4], no characteristic invariant distinguishes representations in the Teichmüller component from those in $C_0$. The Teichmüller component is defined as follows. Using the Klein-Beltrami model of hyperbolic geometry, a hyperbolic structure on $\Sigma$ is a special case of a convex $\mathbb{RP}^2$-structure $\Omega/\Gamma$ where $\Omega$ is the region bounded by a conic. In this case $\Gamma$ is conjugate to a cocompact lattice in $\text{SO}(2,1) \subset \text{SL}(3, \mathbb{R})$. The space $\mathcal{C}(\Sigma)$ of hyperbolic structures (“Teichmüller
space") is a cell of dimension $6(g - 1)$, which is a connected component of $\text{Hom}(\pi, \text{SO}(2, 1))/\text{SO}(2, 1)$. Regarding hyperbolic structures on $\Sigma$ as convex $\mathbb{R}P^2$-structures embeds the Teichmüller space $\mathfrak{T}(\Sigma)$ inside $\mathcal{C}(\Sigma)$. By [5], the space $\mathcal{C}(\Sigma)$ of convex $\mathbb{R}P^2$-structures on a compact surface $\Sigma$ is shown to be diffeomorphic to a cell of dimension $16(g - 1)$ and $\mathfrak{T}(\Sigma)$ embeds $C_1$ as the space of equivalence classes of embeddings of $\pi$ as discrete subgroups of $\text{SO}(2, 1) \subset \text{SL}(3, \mathbb{R})$. Hitchin's component $C_1$ can thus be characterized as the component of $\text{Hom}(\pi, \text{SL}(3, \mathbb{R}))/\text{SL}(3, \mathbb{R})$ containing equivalence classes of discrete embeddings $\pi \to \text{SO}(2, 1)$.

**Theorem A.** Hitchin's Teichmüller component $C_1$ equals the deformation space $\mathcal{C}(\Sigma)$ of convex $\mathbb{R}P^2$-structures on $\Sigma$.

In [5, 3.3] it is shown that the deformation space $\mathcal{C}(\Sigma)$ is an open subset of $\text{Hom}(\pi, \text{SL}(3, \mathbb{R}))/\text{SL}(3, \mathbb{R})$ containing $\mathfrak{T}(\Sigma)$ and hence an open subset of $C_1$. Let

$$\Pi: \text{Hom}(\pi, \text{SL}(3, \mathbb{R})) \to \text{Hom}(\pi, \text{SL}(3, \mathbb{R}))/\text{SL}(3, \mathbb{R})$$

denote the quotient map. Also by [5, 3.2], every representation in $\Pi^{-1}(\mathcal{C}(\Sigma))$ has image Zariski-dense in either a conjugate of $\text{SO}(2, 1)$ or $\text{SL}(3, \mathbb{R})$ itself, and hence by [5, 1.12] $\text{SL}(3, \mathbb{R})$ acts properly and freely on $\Pi^{-1}(\mathcal{C}(\Sigma))$. In particular, the restriction

$$\Pi: \Pi^{-1}(\mathcal{C}(\Sigma)) \to \mathcal{C}(\Sigma)$$

is a locally trivial principal $\text{SL}(3, \mathbb{R})$-bundle. It follows that $\Pi^{-1}(\mathcal{C}(\Sigma))$ is an open subset of $\text{Hom}(\pi, \text{SL}(3, \mathbb{R}))$. Thus Theorem A is a corollary of

**Theorem B.** $\Pi^{-1}(\mathcal{C}(\Sigma))$ is a closed subset of $\text{Hom}(\pi, \text{SL}(3, \mathbb{R}))$.

The rest of the paper is devoted to the proof of Theorem B. Arguments similar to the proof are given at the end of the first chapter of [1] and an analogous statement when $\Sigma$ is a pair-of-pants is proved in [5, §§4.4 and 4.5] (where it is used in the proof of the main theorem). We feel there is a more comprehensive result for compact surfaces with boundary, with a geometric proof.

Assume that $\phi_n$ is a sequence of representations in $\text{Hom}(\pi, \text{SL}(3, \mathbb{R}))$ which converges to $\phi \in \text{Hom}(\pi, \text{SL}(3, \mathbb{R}))$ and that each $\Pi(\phi_n) \in \mathcal{C}(\Sigma)$. Thus for each $n$, there exists a convex domain $\Omega_n \subset \mathbb{R}P^2$ such that $\phi_n: \pi \to \text{SL}(3, \mathbb{R})$ embeds $\pi$ onto a discrete group $\Gamma_n$ acting properly and freely on $\Omega_n$. Furthermore, as discussed in [5, 3.2(1)], each $\Omega_n$ is strictly convex and has the property that the closure $\overline{\Omega_n}$ is a compact subset of an affine patch (the complement of a projective line) in $\mathbb{R}P^2$.

We identify the universal covering of $\mathbb{R}P^2$ with the 2-sphere $S^2$ of oriented directions in $\mathbb{R}^3$. Denote by $p: S^2 \to \mathbb{R}P^2$ the covering projection. The group

$$\text{SL}_\pm(3, \mathbb{R}) = \{ A \in \text{GL}(3, \mathbb{R}) | \det(A) = \pm 1 \}$$

acts on $S^2$ covering the action of $\text{SL}(3, \mathbb{R})$ on $\mathbb{R}P^2 = S^2/\{ \pm 1 \}$. A choice of positive definite inner product on $\mathbb{R}^3$ realizes $S^2$ as the unit sphere in $\mathbb{R}^3$, and $d: S^2 \times S^2 \to \mathbb{R}$ denotes the distance function corresponding to the induced Riemannian metric. The geodesics in $S^2$ are arcs of great circles. If $\Omega \subset \mathbb{R}P^2$ has the property that there exists an affine patch $A \subset \mathbb{R}P^2$ such that $\Omega \subset A$ is convex (with respect to the affine geometry on $A$), then we say that $\Omega$
is properly convex. In that case each component of $p^{-1}(\Omega)$ is convex in the corresponding elliptic geometry of $S^2$ and there exists a sharp convex cone in $\mathbb{R}^3$ whose projectivization equals $\Omega$. We shall also refer to a component of $p^{-1}(\Omega)$ as properly convex. (A sharp convex cone in an affine space $E$ is an open convex domain $\Omega \subset E$ invariant under positive homotheties and containing no complete affine line.)

Since an affine patch is contractible, $p^{-1}(\Omega_n)$ consists of two components each of which maps diffeomorphically to $\Omega_n$. Choose one of the components $\Omega'_n \subset S^2$ for each $n$. Furthermore $\phi_n$ defines an effective proper action of the discrete group $\pi$ on $\Omega'_n$ whose quotient is a convex $\mathbb{RP}^2$-surface homeomorphic to $\Sigma$. Moreover, since $\pi$ is not virtually nilpotent and $\phi_n$ is a discrete embedding for each $n$, the limiting representation $\phi = \lim_{n \to \infty} \phi_n$ is also a discrete embedding (see, e.g., [6, Lemma 1.1]). In particular, the image $\Gamma$ of $\phi$ is torsionfree and not virtually abelian.

Since the space of compact subsets of $S^2$ is compact in the Hausdorff topology, we may (after extracting a subsequence) assume that the sequence $\overline{\Omega'_n}$ converges (in the Hausdorff topology) to a compact subset $K \subset S^2$.

**Lemma 1.** $K$ is invariant under the image $\Gamma = \phi(\pi)$.

*Proof.* Suppose that $k \in K$ and $g \in \pi$. We show that $\phi(g)k \in K$. Let $\varepsilon > 0$. Now $\phi_n(g)$ converges uniformly to $\phi(g)$ on $S^2$; thus there exists $N_1 = N_1(\varepsilon)$ such that

$$d(\phi_n(g)x, \phi(g)x) < \varepsilon/2$$

for $n > N_1$. Indeed the family $\phi_n(g)$ is uniformly Lipschitz for sufficiently large $n$—let $C$ be a Lipschitz constant, i.e.,

$$d(\phi_n(g)x, \phi_n(g)y) \leq C d(x, y)$$

for all $x, y \in S^2$ and $n$ sufficiently large, say $n > N_2$. Since $K$ is the Hausdorff limit of $\overline{\Omega'_n}$, there exist $w_n \in \overline{\Omega'_n}$ such that $w_n \to k$. Thus there exists $N_3 = N_3(\varepsilon)$ such that $d(k, w_n) < \varepsilon/(2C)$ for $n > N_3$. Putting these inequalities together, we obtain

$$d(\phi(g)k, \phi_n(g)w_n) \leq d(\phi(g)k, \phi_n(g)w_n) < \varepsilon/2 + C \varepsilon/(2C) = \varepsilon$$

for $n > \max(N_1, N_2, N_3)$. It follows that $\phi(g)k$ is the limit of $\phi_n(g)w_n \in \overline{\Omega'_n}$. Since the Hausdorff limit of $\overline{\Omega'_n}$ equals $K$, it follows that $\phi(g)k \in K$, as claimed. □

Furthermore each $\overline{\Omega'_n}$ is convex in $S^2$. Since convex sets are closed in the Hausdorff topology, it follows that $K$ is also convex. (See [2] for more details.) There are the following possibilities for $K$ (compare Choi [2]):

1. $K$ is properly convex with nonempty interior.
2. $K$ consists of a single point.
3. $K$ consists of a line segment.
4. $K$ is a great disk (i.e., a closed hemisphere).

We show that only case (1) can arise. The following lemma (whose proof we defer) is used to rule out the last three cases.
Lemma 2. Suppose that $F$ is a nonabelian free group and $h: F \to \text{SL}(2, \mathbb{R})$ is a homomorphism which embeds $F$ onto a discrete subgroup of $\text{SL}(2, \mathbb{R})$. Then there exists $f \in F$ such that $h(f)$ has negative trace.

In cases (2)-(4), there is either a projective line or a point in $\mathbb{RP}^2$ which is invariant under the stabilizer $G$ of $K$. In each of these cases $G$ is conjugate to one of the subgroups of $\text{SL}(3, \mathbb{R})$ consisting of matrices

\[
\begin{pmatrix}
* & 0 & 0 \\
* & * & * \\
* & * & *
\end{pmatrix}, \quad \begin{pmatrix}
* & * & * \\
0 & * & * \\
0 & * & *
\end{pmatrix}.
\]

In both cases, there is a homomorphism $\rho: G \to \text{SL}(2, \mathbb{R})$ such that if $g \in [G, G]$ then

\[\text{tr}(g) = 1 + \text{tr}(\rho(g)).\]

(We take $g$ to lie in the commutator subgroup so as to assume that the $(1,1)$-matrix entry and the determinant of the $(2 \times 2)$-block are both 1.)

We suppose that $\phi_n$ is a sequence as above converging to $\phi$. Since $\phi$ is a discrete embedding, apply Lemma 2 to the restriction $h$ of $\rho \circ \phi$ to $F = [\pi, \pi]$. We deduce that there exists $\gamma \in \pi$ such that $\text{tr}(\phi(\gamma)) < 1$. However, as discussed in [5, 3.2(3)], every $1 \neq \gamma \in \pi$ has the property that $\phi_n(\gamma) \in \text{SL}(3, \mathbb{R})$ has positive eigenvalues; in particular, $\text{tr} \phi_n(\gamma) > 3$. Since $\phi_n \to \phi$, it follows that $\text{tr} \phi(\gamma) \geq 3$, a contradiction.

Thus only case (1) is possible: $K$ is properly convex with interior $\Omega$. Then $\Gamma$ acts isometrically with respect to the Hilbert metric on $\Omega$. Since $\Gamma$ is discrete, torsionfree, and acts properly on $\Omega$, the quotient $\Omega/\Gamma$ is a closed surface. Since $\Gamma$ is not virtually abelian, $\Omega$ is not a triangular region and by [8] (see also [5, 3.2]) it follows that $\Omega/\Gamma$ is a convex $\mathbb{RP}^2$-manifold homeomorphic to $\Sigma$. This concludes the proof of Theorem B, assuming Lemma 2.

Proof of Lemma 2. By passing to a subgroup of $F$ we may assume that the quotient of the hyperbolic plane by the image of $h(F)$ under the quotient homomorphism $\text{SL}(2, \mathbb{R}) \to \text{PSL}(2, \mathbb{R})$ is a complete surface which is homeomorphic to a pair-of-pants $P$ (a sphere minus three discs). Let $f_1, f_2, f_3$ be elements of $\pi_1(P) \subset F$ corresponding to the three components of $\partial P$. We choose elements $\widehat{h(f_j)} \in \text{SL}(2, \mathbb{R})$ ($j = 1, 2, 3$) so that $\text{tr}(\widehat{h(f_j)}) > 2$ (equivalently, $\widehat{h(f_j)}$ lies in a hyperbolic one-parameter subgroup of $\text{SL}(2, \mathbb{R})$). Now $f_1f_2f_3 = 1$ in $F$ but

\[
\text{tr}(\widehat{h(f_1)}h(\widehat{f_2})h(\widehat{f_3})) = \left(-1\right)^{\chi(P)} = -1
\]

(since the relative Euler class of the representation equals $-1$; compare the discussion in [4, §4]). Since each $h(f_j)$ is hyperbolic, an odd number of $f_j$ must satisfy $\text{tr}(h(f_j)) < 2$; in particular, at least one $f \in F$ satisfies $\text{tr}(h(f)) < 0$. □

References

2. ———, Compact $\mathbb{RP}^2$-surfaces with convex boundary I: $\pi$-annuli and convexity (submitted).

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