COUNTEREXAMPLES FOR BROWN-PEDERSEN’S CONJECTURE IN “C*-ALGEBRAS OF REAL RANK ZERO”

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Abstract. We give nonseparable C*-algebras which are counterexamples for Brown-Pedersen’s conjecture in “C*-algebras of real rank zero”.

1. Introduction

Let $A$ be a unital C*-algebra and $A_{sa}$ be the set of all selfadjoint elements in $A$. For nonnegative integer $n$, consider the set of left $n$-tuples

$$L_n(A) = \left\{(a_0, a_1, \ldots, a_n) \in A_{sa}^{n+1}; \sum_{k=0}^{n} Aa_k = A \right\}.$$

The real rank of $A$, denoted $RR(A)$, is the least integer $n$ such that $L_n(A)$ is dense in $A_{sa}^{n+1}$.

If $A$ is nonunital, its real rank is defined by $RR(\tilde{A})$, where $\tilde{A} = A \oplus \mathbb{C}$.

In [1] Brown and Pedersen defined and studied this concept and especially characterized C*-algebras of real rank zero. The following three conjectures are given:

1. $RR(M(A)) = 0$ for every AF-algebra $A$,
2. $RR(M(A)) = 0$ for every C*-algebra $A$ for which $RR(A) = 0$ and $K_1(A) = 0$, and
3. if $A$ is a C*-algebra with $RR(A) = 0$, then $RR(M(A)/A) = 0$,

where $M(A)$ denotes the multiplier C*-algebra of $A$ and AF-algebras mean approximately finite-dimensional C*-algebras. Recently, conjecture (1) has been proved by Lin [4] in the $\sigma$-unital case.

In this note, we present counterexamples for these three conjectures, although those C*-algebras are non-$\sigma$-unital.

We refer the reader to [1] for results about real ranks.

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2. Counterexamples

First we introduce a normal space $Y$ with the covering dimension ($= \dim Y$) $\neq 0$, which is Dowker's example [2, 6.2.20].

Let $Q$ denote the set of all rational numbers in the interval $I = [0, 1]$. By taking

$$x \sim y \text{ if and only if } |x - y| \in Q$$

we define an equivalence relation on $I$. Then the family of all equivalence classes has cardinality $\aleph_1$. Choose a subfamily of cardinality $\aleph_1$ which does not contain the equivalence class $Q$, and let us arrange its members into a transfinite sequence $Q_0, Q_1, \ldots, Q_a, \ldots$, $a < \omega_1$, where $a$ is an ordinal number and $\omega_1$ is the smallest uncountable ordinal number of cardinality $\aleph_1$.

For every $a < \omega_1$ the set $S_a = I \setminus \bigcup_{y \geq a} Q_y$ is zero-dimensional, that is, a nonempty $T_1$-space, and has a base consisting of open and closed sets. Let $X$ be the space of all ordinal numbers $\leq \omega_1$. Consider on $X$ the topology generated by the base consisting of all segments $(y, x] = \{z \in X; y < z \leq x\}$ ($y < x \leq \omega_1$) and the one-point $\{0\}$, where $0$ is the order type of the empty set. We know that $X$ is a compact Hausdorff space. For every $a < \omega_1$, $X_a = \{y \in X; y \leq a\}$ is open and closed in $X$; hence, the Cartesian product $X_a \times S_a$ is a zero-dimensional metrizable space. Let $Y_a = \bigcup_{y \leq a} (\{y\} \times S_y)$ and $Y = \bigcup_{a < \omega_1} Y_a$. Since $Y_a = Y \cap (X_a \times I)$, it is open and closed in $Y$ and a zero-dimensional metrizable space; hence, $\dim Y_a = 0$. But we know $\dim Y \geq 0$ due to Dowker [2, 6.2.20].

Since $Y$ is not locally compact, we modify $Y$ to a locally compact space by Isbell's result [3, Chapter VI.14].

Let $W$ be the set $X - \{\omega_1\}$, and let $p: Y \to W$ be the first coordinate projection. Let $\beta Y$ be the Stone–Čech compactification of $Y$. Then we have an extension map $\beta p: \beta Y \to \beta W$, and let $Z$ be $(\beta p)^{-1}(W)$. It is obvious that $Z$ is locally compact. Since $Y \subset Z \subset \beta Y$, $\beta Z = \beta Y$ and $\dim \beta Z = \dim \beta Y \geq 0$. Let $Z_a = Z \cap (X_a \times I)$, then $Z = \bigcup_{a < \omega_1} Z_a$. Since the small inductive dimension of $Z$ ($= \ind Z$) $= 0$, $\ind Z_a = 0$ by [2, Theorem 7.1.1] and $\dim Z_a = 0$ by [2, Theorem 7.1.11].

Let $C_0(Z)$ be the commutative $C^*$-algebra of all complex continuous functions of $Z$ vanishing at infinity. By the previous construction for $Z$, $C_0(Z)$ is an inductive limit of $C_0(Z_a)$. As $\dim Z_a = 0$, $C_0(Z_a)$ is AF, and hence $C_0(Z)$ is AF.

We get counterexamples for the previous three conjectures in the nonseparable case.

**Theorem.** Let $n$ be any positive integer $\geq 1$. Consider the following $C^*$-exact sequence:

$$0 \to C_0(Z) \otimes M_n(\mathbb{C}) \to M(C_0(Z) \otimes M_n(\mathbb{C})) \to M(C_0(Z) \otimes M_n(\mathbb{C}))/C_0(Z) \otimes M_n(\mathbb{C}) \to 0.$$

Then $\RR(C_0(Z) \otimes M_n(\mathbb{C})) = 0$, $\RR(M(C_0(Z) \otimes M_n(\mathbb{C}))/C_0(Z) \otimes M_n(\mathbb{C})) \neq 0$, and $\RR(M(C_0(Z) \otimes M_n(\mathbb{C}))) \neq 0$.

**Proof.** Since $C_0(Z)$ is AF, so is $C_0(Z) \otimes M_n(\mathbb{C})$ for any integer $n \geq 1$, and $\RR(C_0(Z) \otimes M_n(\mathbb{C})) = 0$ by [1, Proposition 3.1].
It is obvious that $M(C_0(Z) \otimes M_n(\mathbb{C})) = C(\beta Z) \otimes M_n(\mathbb{C})$. Since $\dim Z = \dim \beta Z \geq 0$ (see [2, Theorem 7.1.17]), $RR(C(\beta Z)) = \dim Z \geq 0$ from [1, Proposition 1.1]. Moreover, we know $RR(C(\beta Z) \otimes M_n(\mathbb{C})) \geq 0$ from [1, Corollary 2.8].

Suppose that $RR(C(\beta Z \setminus Z)) = 0$. By [1, Proposition 1.1] $\dim \beta Z \setminus Z = 0$, and $C(\beta Z \setminus Z)$ is AF. Since every projection in $C(\beta Z \setminus Z)$ lifts to a projection in $C(\beta Z)$, we know $RR(C(\beta Z)) = 0$ from [1, Theorem 3.14], which is a contradiction. Hence, $RR(C(\beta Z \setminus Z)) \geq 0$. As $M(C_0(Z) \otimes M_n(\mathbb{C}))/C_0(Z) \otimes M_n(\mathbb{C}) = C(\beta Z \setminus Z) \otimes M_n(\mathbb{C})$, we know

$$RR(M(C_0(Z) \otimes M_n(\mathbb{C}))/C_0(Z) \otimes M_n(\mathbb{C})) \geq 0$$

as in the above argument. □

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