AN EXAMPLE OF A CARATHÉODORY COMPLETE
BUT NOT FINITELY COMPACT ANALYTIC SPACE

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Abstract. An analytic space is given which is $c_X$-complete but not $c_X$-finitely compact.

1. Introduction

If $X$ is a (connected) complex analytic space, the Carathéodory pseudodistance $c_X$ on $X$ is given by

$$c_X(x, y) = \sup\{\omega(f(x), f(y)) : f \in \mathcal{O}(X, \Delta)\},$$

where $\mathcal{O}(X, \Delta)$ denotes the set of all holomorphic mappings from $X$ to the open unit disc $\Delta \subset \mathbb{C}$ and where $\omega$ is the hyperbolic (Poincaré) distance on $\Delta$. Now let us consider an analytic space $X$ for which $c_X$ is a distance defining the topology of $X$ (cf. [2, 7]). If the $c_X$-balls are relatively compact in $X$, i.e., $X$ is finitely compact with respect to $c_X$, then it is clear that $X$ is $c_X$-complete. The problem of the equivalence of these two notions has been raised by Kobayashi [4] (see also [1]). A positive answer for plane domains was given by Selby [5] and Sibony [6]. (We thank C. J. Earle for showing us the paper of Selby.) The purpose of this note is to construct an analytic space $X$, $c_X$-complete but not finitely compact with respect to $c_X$. The case of domains in $\mathbb{C}^n$, $n > 1$, still remains open.

2. Construction of the example

For every integer $n > 0$ let

$$p^n_k := \left(1 - \frac{1}{n + 1}\right) \exp\left(\frac{2\pi ik}{n + 1}\right) \in \Delta, \quad 0 \leq k \leq n.$$  

We construct a connected analytic space $X$ in the following way:

Let $(D_j)_{j=0}^\infty$ be a sequence of copies of $\Delta$, i.e., $D_j := \Delta$, and let $q^n_0 \in D_0$ be defined by $q^n_0 := p^n_0$, $n > 0$, $0 \leq k \leq n$. Moreover, for every $n > 0$, let $x^n_k \in D_n$ be given by $x^n_k := p^n_k$, $0 \leq k \leq n$. 

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Then the analytic space $X$ is obtained by patching together $D_n$ and $D_0$, $n > 0$, by identifying the points $q^n_k$ and $x^n_k$, $0 \leq k \leq n$.

Topologically, $X$ is the quotient of $\bigcup_{n=0}^{\infty} D_n$ by the equivalence relation $\mathcal{R} : q_k^n \mathcal{R} x_k^n$, $n > 0$, $k = 0, \ldots, n$. The analytic structure of $X$ is defined by gluing $D_0$ and $D_n$, $n > 0$, transversally at $q_k^n \sim x_k^n$, $0 \leq k \leq n$.

It is clear that $X$ is a connected one-dimensional reducible analytic space. A holomorphic function $f$ on $X$ can be identified with a family $(f_n)_{n \geq 0}$ of holomorphic functions $f_n$ on $D_n$, $n \geq 0$, such that $f_n(x^n_k) = f_0(q^n_k)$, $n > 0$, $0 \leq k \leq n$.

Let $O_n$ be the image in $X$ of the origin $O$ of $D_n$, $n \geq 0$.

**Theorem.** (a) The Carathéodory pseudodistance $c_X$ is a distance on $X$, it defines the topology of $X$, and the space $X$ is $c_X$-complete.

(b) There exists $r > 0$ such that the Carathéodory ball $B_{c_X}(O_0, r)$ is not relatively compact in $X$.

Observe that by the Hurwitz theorem we have the following:

**Lemma 1.** Let

$$f_n(\lambda) := \prod_{k=0}^{n} \frac{\lambda - p^n_k}{1 - p^n_k \lambda}, \quad \lambda \in \Delta.$$ 

Suppose that $f_{n_j} \to f$ locally uniformly on $\Delta$. Then $|f(0)| = 1/e$ and $f$ is without zeros on $\Delta$.

The natural isomorphism $i_n : D_n \to \Delta$ induces a holomorphic mapping $p : X \to D_0$.

**Lemma 2.** Let $K \subset \subset \Delta$, and let $N_0$ be an integer such that $p^n_k \notin K$, $n \geq N_0$, $0 \leq k \leq n$. Denote by $X_{N_0}$ the union of the images in $X$ of the $D_n$, $n \geq N_0$. Then there exists a constant $C > 0$ such that, whenever $x, y \in X_{N_0}$, $x \neq y$, with $p(x) = p(y) \in K$, then $c_X(x, y) > C$.

**Proof.** Suppose that $c_X(x_\nu, y_\nu) \to 0$, where $x_\nu, y_\nu \in X$, $x_\nu \neq y_\nu$, and $p(x_\nu) = p(y_\nu) \in K$. Since bounded holomorphic functions on $X$ separate points of $X$, we can assume that $x_\nu \in D_{n(\nu)}$ with $n(\nu) \to \infty$.

Observe that $F_n$ with $F_n(x) := 0$, $x \in D_m$, $m \neq n$, and $F_n(x) := f_n(x)$, $x \in D_n$, is holomorphic on $X$. Therefore, since $c_X(x_\nu, y_\nu) \to 0$, we get that $f_{n(\nu)}(x_\nu) \to 0$, which contradicts Lemma 1. $\Box$

Now, we pass to the proof of the theorem.

**Proof.** Assertion (a) follows directly from Lemma 2. In order to prove (b), let $g : X \to \Delta$ be holomorphic with $g(O_0) = 0$. Define on $\Delta$

$$\varphi_n(\lambda) := \frac{1}{2} (g(\lambda_n) - g(\lambda_0)),$$

where $\lambda_k \in X$ denotes the image in $X$ of $\lambda \in D_k$, $k \geq 0$.

It is clear that $\varphi_n \in \mathcal{O}(\Delta, \Delta)$ and that $\varphi_n(p^n_k) = 0$, $0 \leq k \leq n$. Therefore, by the Schwarz Lemma, we obtain that

$$|\varphi_n(0)| \leq \left(1 - \frac{1}{1 + n}\right)^{n+1},$$
which gives $|g(O_n)| \leq 2(1 - \frac{1}{1+n})^{n+1}$. Hence $O_n \in B_{c_X}(O_0, r)$, $r := \omega(0, \frac{4}{3})$, if $n \gg 1$. □

Observe that $B_{c_X}(O_0, r)$ is disconnected and has relatively compact components. Therefore the closure of $B_{c_X}(O_0, s)$ is not equal to the closed Carathéodory ball $\{y \in X: c_X(O_0, y) \leq s\}$ for some $s$ (cf. [3]).

In order to conclude we recall that, so far, there is no example of an analytic space satisfying (a) of the theorem but which is not $H^\infty$-convex.

REFERENCES