A REMARK ON CURVES COVERED BY COVERINGS

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ABSTRACT. Let \( f : C' \rightarrow C \) be a covering between two smooth irreducible projective curves. Let \( p \) be a prime number. If \( C' \) is a covering of degree \( p \) of a curve of genus \( h \) and if \( g(C) \geq \left( \frac{p}{2} \right)(h+3)+h+3-2p \), then \( C \) is a covering of degree \( p \) of a curve of genus at most \( h \).

Let \( C \) be a smooth irreducible projective curve of genus \( g \) defined over \( \mathbb{C} \).

Definition. We say that \( C \) is of type \((d; h)\) if there exists a covering \( \pi : C \rightarrow E \) with \( \deg(\pi) = d \) and \( g(E) \leq h \).

Statement \( S(d; h; g) \). If \( f: C' \rightarrow C \) is a morphism with \( g(C) = g \) and if \( C' \) is of type \((d; h)\), then \( C \) is of type \((d; h)\).

Statement \( S(d; h; g) \) is proved for the cases \( d = 2 \) (see [1, 2]), \( d = 3 \) (see [1]), and \( d = 4, \ g \neq 7 \) (see [1]). For \( h = 0 \) it is almost trivial (see, e.g., [4]).

In this paper we prove

Theorem. Let \( p \) be a prime number. Statement \( S(p; h; g) \) holds for \( g \geq \left( \frac{p}{2} \right)(h+3)+h+3-2p \).

Remarks (from the referee). (1) There is an important difference between Statement \( S(d; h; g) \) in this paper and in [1]. As a matter of fact, in [1] a curve \( C \) is said to be of type \((d; h)\) if \( \deg(\pi) \leq d \) (inequality instead of equality).

(2) The referee informed me about D. Abramovich's Ph.D. Thesis, Subvarieties of Abelian varieties and of jacobians of curves, of which Abramovich sent me a copy. Amongst a lot of other interesting results, it contains again results concerning the topic studied in this paper. Also, from a letter, it is clear to me that he knows our theorem already.

If \( d \) is not a prime number then the situation is more complicated. Assume \( d = n = an' \) for some \( a, n' \in \mathbb{Z}_{\geq 2} \) and assume there exists a diagram

\[
\begin{array}{ccc}
C_1' & \longrightarrow & C_1 \\
\downarrow \pi_1 & & \\
E & \end{array}
\]

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with \( \deg(\pi_1) = n' \), \( g(E) = h \), and \( g(C_1) = g_1 \) such that \( C_1 \) is not of type \((n''; h)\) for some \( n'' < n' \). Take a covering \( \tau: C \to C_1 \) of degree \( a \). Assume there exists a covering \( \mu: C \to E_1 \) with \( \deg(\mu) \leq n \) and \( g(E_1) \leq h \). Consider \((\tau; \mu): C \to C_1 \times E_1\), and let \( C'' \) be the normalization of the image of \((\tau; \mu)\). One obtains a diagram of coverings

\[
\begin{array}{ccc}
  C & \xrightarrow{\nu} & C'' \\
  \tau \downarrow & & \gamma \downarrow \\
  C_1 & \xrightarrow{\mu} & E_1
\end{array}
\]

It is well known (see, e.g., [3, §1, (iii)]) that \( g(C'') \leq (a-1)(n-1) + ag_1 + nh \). So if we take \( g(C) \) large enough then \( \nu \) is nontrivial. Also \( \gamma \) is nontrivial, otherwise \( \varepsilon \) gives us that \( C_1 \) is of type \((n''; h)\) for some \( n'' < n' \). Choosing suited ramification in the covering \( \tau \) (e.g., a fibre of type \((a-1)Q_1 + Q_2\) suffices), this becomes impossible. Let \( \widetilde{C'} \) be the normalization of an irreducible component of \( C_1^{'} \times C_1 C \). Taking a covering \( C' \to \widetilde{C'} \) if necessary, we obtain the following situation

\[
\begin{array}{ccc}
  C' \xrightarrow{\pi} C \\
  \downarrow & & \\
  E
\end{array}
\]

with \( \deg(\pi) = n \), \( g(E) \leq h \), but \( C \) is not of type \((n; h)\). Moreover, we can take \( g(C) \) arbitrarily large.

In [1] Abramovich and Harris prove the existence of a diagram

\[
\begin{array}{ccc}
  C_1^{'} \xrightarrow{\pi} C_1 \\
  \downarrow & & \\
  E
\end{array}
\]

with \( g(C_1) = 5 \), \( \deg(\pi) = 3 \), \( g(E) = 2 \), and \( C_1 \) is not of type \((n''; 2)\) for \( n'' = 2 \) or 3. Hence for each \( a \in \mathbb{Z}_{\geq 2} \) there exists no \( g_0 \) such that \( S(3a; 2; g) \) holds for \( g \geq g_0 \).

**Proof of the theorem.** Consider the situation

\[
\begin{array}{ccc}
  C' \xrightarrow{f} C \\
  \downarrow & & \\
  E
\end{array}
\]

with \( \deg(\pi) = p \), \( g(E) = h \), and \( g(C) = g \geq (\text{deg}(\mathcal{L}))(h+3) + h + 3 - 2p \). For \( n \in \mathbb{Z}_{\geq 2} \), \( D \in E^{(h+n)} \), and \( P \in E \) general, the linear systems \( |D - P| \) and \( |D| \) on \( E \) are without fixed points. Using the fact that \( f_* \) preserves linear equivalence (see [4]) also \( L := |f_* \mathcal{L}| \) and \( |L - f_* \pi^*(P)| = |f_* \pi^*(D - P)| \) have no fixed points on \( C \). In particular,

\[
(*) \quad \dim(L) \geq \dim(|L - f_* \pi^*(P)|) + 1.
\]
Assume we have equality in (*). Let \( \phi: C \to \mathbb{P}^r \) be the morphism associated to \( L \), let \( C_1 \) be the normalization of \( \phi(C) \), and let \( \rho: C \to C_1 \) be the induced covering. Equality in (*) implies that \( \deg(\rho) = p \) and for each \( P \in E \) the divisor \( f_*\pi^*(P) \) is a fibre of \( \rho \). It follows that the morphism \( (\pi; \rho \circ f): C' \to E \times C_1 \) factors through \( E \). Hence \( E \) dominates \( C_1 \), so \( g(C_1) \leq h \). This implies that \( C \) is of type \( (p; h) \).

So now, assume we have

\[
\dim(L) \geq \dim(|L - f_*\pi^*(P)|) + 2.
\]

Because \( \deg(\rho) < p \) now, we have that \( \rho \) is an isomorphism (here we use that \( p \) is a prime number). If \( p = 2 \) then for \( D \in E^{(h+1)} \) general one finds that

\[
\dim(|f_*\pi^*(D) + f_*\pi^*(P)|) \geq \dim(|f_*\pi^*(D)|) + \deg(f_*\pi^*(P)).
\]

This implies that \( |f_*\pi^*(D)| \) is nonspecial, hence \( 1 \leq 2h+2-g \), i.e., \( g \leq 2h+1 \). Since \( g > 2h+1 \), we obtain a contradiction. For \( p \neq 2 \) we are going to use Castelnuovo theory in order to obtain a similar situation. The use of Castelnuovo theory is inspired by §3 in [1].

Take \( D_1 \in E^{(h+2)} \) general and \( D_2 \in E^{(h+n)} \) \((n \geq 2)\) general. Take \( P \in E \) general and write \( f_*\pi^*(P) = P_1 + P_2 + \cdots + P_p \). Since \( L_i = |f_*\pi^*(D_i)| \) is simple, we obtain that there exists \( E_i \in L_i \) with \( \text{cd}(E_i; f_*\pi^*(P)) = P_1 \). (For two effective divisors \( E_1 \) and \( E_2 \) on \( C \), one has \( E_0 = \text{cd}(E_1; E_2) \) if \( E_i \geq E_0 \) and \( \text{Supp}(E_1 - E_0) \cap \text{Supp}(E_2 - E_0) = \emptyset \).) Consider \( |E_1 + E_2 + P_3 + \cdots + P_p| \supset (|E_1 + E_2| + P_3 + \cdots + P_p) \cup (|E_1 + P_2 + \cdots + P_p| + (E_2 - P_2)) \). Also \( |E_1 + P_2 + \cdots + P_p| = |f_*\pi^*(D_1 + P) - P| \) has no fixed points, so \( |E_1 + E_2 + P_3 + \cdots + P_p| \) has no fixed points. Also \( |E_1 + E_2 + P_3 + \cdots + P_p| \supset |E_2 + P_1 + \cdots + P_p| + (E_1 - P_1) \), hence \( |E_1 + E_2 + P_3 + \cdots + P_p| \) has no fixed points. Finally \( |E_1 + E_2 + P_1 + \cdots + P_p| = |f_*\pi^*(D_1 + D_2 + P)| \) has no fixed points. This implies, for \( D \in E^{(2h+4+n)} \) \((n \geq 0)\) and \( P \in E \) general, that one has

\[
\dim|f_*\pi^*(D + P)| \geq \dim|f_*\pi^*(D)| + 3;
\]

there exist \( E_i \in |f_*\pi^*(D + P)| \) for \( i = 1, 2 \)

with \( \text{cd}(E_1; f_*\pi^*(P)) = P_1, \text{cd}(E_2; f_*\pi^*(P)) = P_1 + P_2 \).

Assume \( p > 3 \). Take \( D_1 \in E^{(2h+5)} \) general, \( D_2 \in E^{(h+2)} \) general, and \( P \in E \) general, and write \( f_*\pi^*(P) = P_1 + \cdots + P_p \). Take \( E_1, E_2 \in |f_*\pi^*(D_1)| \) with \( \text{cd}(E_1; f_*\pi^*(P)) = P_1, \text{cd}(E_2; f_*\pi^*(P)) = P_1 + P_2 \), and take \( E_3 \in |f_*\pi^*(D_2)| \) with \( \text{cd}(E_3; f_*\pi^*(P)) = P_3 \). As before, for \( D \in E^{(3h+7)} \) and \( P \in E \) general, one has

\[
\dim|f_*\pi^*(D + P)| \geq \dim|f_*\pi^*(D)| + 4;
\]

there exist \( E_i \in |f_*\pi^*(D + P)| \) for \( i = 1, 2, 3 \)

with \( \text{cd}(E_i; f_*\pi^*(P)) = P_1 + \cdots + P_i \).

Continuing in this way, one finds for \( D \in E^{((p-1)(h+3)-2)} \) and \( P \in E \) general that

\[
\dim|f_*\pi^*(D + P)| \geq \dim|f_*\pi^*(D)| + p.
\]

This is only possible if \( |f_*\pi^*(D)| \) is nonspecial on \( C \). But \( \deg(f_*\pi^*(D)) = p(p-1)(h+3) - 2p \) and \( \dim(|f_*\pi^*(D)|) \geq 1 + 2(h+3) + 3(h+3) + \cdots + (p-1)(h+3) + (p-1)h - 2 \). So \( (p-1)(h+3) - h - 2 \leq p(p-1)(h+3) - 2p - g \), hence \( g \leq (p-1)(h+3) - 2p + h + 2 \). This gives us a contradiction.
Remark. Using the arguments of [1], the situation is related to the existence of an abelian variety $A$ of dimension $h$ contained in $C^{(p^h)}$. Then Theorem 2 in [1] gives us a bound of order $p^2h^2$, which is worse than ours.

REFERENCES